# MULTISENSORY ALGEBRA THROUGH CONCRETE TO REPRESENTATIONAL TO ABSTRACT INSTRUCTION FOR MIDDLE SCHOOL STUDENTS WITH LEARNING DIFFICULTIES

By

BRADLEY STEVEN WITZEL

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MULTISENSORY ALGEBRA THROUGH CONCRETE TO REPRESENTATIONAL TO ABSTRACT INSTRUCTION FOR MIDDLE SCHOOL STUDENTS WITH LEARNING DIFFICULTIES

By

Bradley Steven Witzel

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Chair: Cecil D. Mercer
Department: Special Education

Learning algebra is a complex task, especially for students with math learning disabilities and difficulties. A few publishers and researchers have produced curriculum programs to help low-performing students with initial algebra instruction, but none of these programs have been shown to help students past solving inverse operations. This research examined a newly developed concrete-to-representational-to-abstract (CRA) algebra program that does generalize to more complex equations. Thirty-four matches of students with disabilities and students at risk for disabilities were obtained from the 358 students who participated in this program across four separate middle schools in an urban county in Florida. Students were matched according to pretest score, standardized math test scores, the same current math teacher, similar age and same grade level, and semester averages. All 358 students participated in the 19-lesson curriculum that took students from reducing algebraic expressions to solving multiple step single variable

transformational algebra equations. Teachers taught one-half of the matches using traditional instruction while the other half were taught using the CRA program. Students' scores were compared across pretests, posttests, and follow-up tests. Students who received CRA instruction showed statistically significant higher growth in transformational equations than traditionally taught peers from pretest to posttest and follow-up test scores, $F(2, 66) = 13.888$, $p = 0.000$). Post hoc $t$-tests revealed that the students who learned through this newly developed CRA algebra program outperformed their traditionally taught peers on both the posttest and the 3-week follow-up test. The positive results from this research raise the possibility of teaching algebra to students with learning problems through well- constructed, hands-on, and pictorial instruction first.

# CHAPTER 1
# INTRODUCTION

For the past two decades, America's public schools have come under attack for students' poor performance in mathematics. Starting with *A Nation at Risk* (National Commission on Excellence in Education [NCEE], 1983), the U.S. was depicted as a declining nation for mathematics education. Findings from the Third International Mathematics and Science Study (TIMSS) (National Center for Educational Statistics [NCES], 1999) indicated that U.S. students compared poorly on measures of math achievement to students from 40 other countries, and this performance became more evident as students entered secondary school. While U.S. fourth-grade students were above the international average in math and science achievement, eighth-grade students were not. In fact, by the time U.S. students reached eighth grade, they were below average. By the final year of secondary school, U.S. students were among the lowest in the world in both math and science achievement. Interestingly, results from TIMSS (NCES, 1999) indicated that the seventh-grade math curriculum for most nations was equivalent to the U.S. ninth-grade curriculum.

Prior to TIMSS, U.S. educators were concerned with poor mathematics achievement nationally, specifically with the drop in mathematics achievement from fourth to twelfth grades (NCES, 1999). In fact, heightened concern over the past two decades resulted in a variety of reform movements in mathematics education. These reform movements were spurred by *A Nation at Risk* (NCEE, 1983) and led to an

increase in number and difficulty of mathematics course requirements for graduation (NCES, 1996) including assessments given to high school seniors to ensure they have sufficiently learned material before graduation.

In a similar vein, state boards of education have also instituted a list of graduation requirements and assessments that students must pass to receive a high school diploma. Some states now include course requirements for graduation such as passing algebra (NCES, 1996). Georgia, Louisiana, New York, North Carolina, South Carolina, Texas, and Virginia even use performance-based graduation examinations including algebra knowledge that students must pass to receive a standard diploma (Ysseldyke et al., 1998). In addition, North Carolina and Texas instituted end-of-course algebra tests as a means of assessing students beyond grade reports. Although students may receive an "A" grade in their algebra course, they still must receive a passing score on the statewide exam at the end of the academic year. State departments use these statewide assessments to ensure that students understand needed mathematical concepts and to ensure that classroom teachers be accountable for providing adequate instruction.

While states have focused on increasing requirements, the National Council of Teachers of Mathematics (NCTM) recognized the need to increase the quality of mathematics courses and consequently published standards to improve mathematics curriculum and instruction (Rivera, 1996). The NCTM (1989) standards list a priority in algebra as improving students' ability to represent authentic word problems involving variable quantities and equations, inequalities, and matrices. They also stress a student's ability to use tables and graphs as tools to interpret equations and inequalities.

While math is an important curricular component of a high school education, raising graduation requirements and standards may not necessarily increase student achievement, particularly for students with disabilities. Although Goals 2000: Educate America Act of 1993 calls for a 90% graduation rate (Lewis, 1999), neither students without disabilities nor students with disabilities are successfully completing graduation exams. In a review of 115 state reports on education, Ysseldyke et al. (1998) found a difference in graduation test success rates for students with disabilities compared to rates for students without disabilities. In Georgia, 81% of all students passed graduation examinations on their first attempt, which is nearly twice the 45% rate for students with special needs. Similarly, in Louisiana, the comparison of graduation rates for students without disabilities to students with special needs is 78% to 49%, respectively (Ysseldyke et al., 1998). Nationally, students are not matching the desired progress set by Goals 2000, particularly students with disabilities. Moreover, Hoffer (1997), in an analysis of the National Education Longitudinal Study of 1988, found that requiring more mathematics courses had no effect on the probability of dropping out or achieving more. If the goal of the Educate America Act is to increase the graduation rate and the goal of U.S. education is to improve student understanding, then increasing the number of courses or implementing high stakes assessment may not be the answer.

In an attempt to understand why U.S. students were not comparing favorably with students in other countries, researchers at the National Center for Educational Statistics (NCES, 1999) found that U.S. achievement was not related to the amount of homework students completed or to the length of instructional time. It could be that quality of instruction is a salient variable in improving math achievement. In the U.S., a problem in

mathematics instruction is curriculum coverage. There is a disproportion between the large number of lessons in a typical mathematics textbook and the smaller number of school days to deliver the lessons (Mercer & Mercer, 1998). Such incongruity is alarming, since teachers base 75% to 95% of the instructional sequence on school-issued textbooks (Tyson & Woodward, 1989). The large number of lessons may cause teachers to move through assignments at a fast pace, not allowing students to master the concepts needed to comprehend future assignments. It is only logical that as lessons grow increasingly difficult throughout the year, students who have not mastered early concepts will fall behind, slowing their future understandings.

A fast-paced math curriculum and increasing political pressure on mathematics achievement testing have caused difficulties with both acquisition and retention of mathematical knowledge for students with learning disabilities (Miller & Mercer, 1997). Miller and Mercer found acquisition and retention of mathematical concepts and facts as two difficult tasks for students with learning disabilities. While normally achieving students learn math concepts in a steadily increasing pattern, students with learning disabilities acquire skills in a broken sequence and have lower retention rates than their nondisabled peers (Cawley, Parmar, Yan, & Miller, 1996; Geary, Hoard, & Hamson, 1999). To discover what difficulties exist for secondary students with learning disabilities, Miles and Forcht (1995) studied the cognitive strategies of students. They found that students with learning disabilities begin to develop difficulty at the abstract level. Difficulty at the abstract level means that students with learning disabilities can conceptually understand math principles using representations (e.g., pictures to display math concepts), but they have difficulty computing mathematics using numbers only.

Difficulty with abstract numbers may explain these students' consistently low scores on graduation exams.

Devlin (2000) stated that for students to understand abstract concepts more easily, it is important for them to have a concrete understanding of those concepts first. One way to simplify students' understanding of abstract concepts is to transform such complex concepts into concrete manipulations and pictorial representations. Instruction incorporating concrete manipulations and pictorial representations is the concrete-to-representational-to-abstract sequence of instruction (CRA). While much research on CRA focused on the effectiveness with arithmetic instruction (Miller & Mercer, 1993), recently more researchers attempted to design CRA models for algebra instruction (Borensen, 1997; Maccini & Hughes, 2000).

Maccini and Hughes (2000) demonstrated the effectiveness of their CRA model with secondary education students with learning disabilities in representing and solving single variable algebra word problems. However, their model does not generalize beyond one-step single variable equations. For example, $X + 3 = 5$ is easily represented in their model. Similarly, $5X = 15$ is also easily represented in their model, but the representations are not accurate. By the students representing the variable X with a colored cube, such as yellow, the coefficient is misrepresented. Instead of thinking five cubes is 5X, mathematically, five cubes should be $X^5$ in more complex equations. In a practical sense, when the student who learned that five yellow cubes represent the abstract 5X, how is he or she going to represent $X^5$? By creating incorrect models for representational and concrete stages, it is likely that students may benefit from a lower conceptual learning. However, when the students are confronted with more complex

equations, the incorrect model may likely cause confusion and oversimplification of the concept. The incorrect model used by Maccini and Hughes (2000) represents a serious weakness in their CRA model and thus the long-term effectiveness for students.

The purpose of this research was to test the effectiveness of a new CRA model that was capable of representing more complex equations. Effectiveness was judged according to posttest and a 3-week follow-up measure for students with learning disabilities and those who were at risk for failure in secondary mathematics. The scores of the students who were taught using the CRA model were compared to scores of matched peers taught using abstract forms of instruction. Comparison was statistically measured using an analysis of variance computation similar to a dependent sample t-test measure. To reduce error and increase power, students with an at-risk concern or disability label were matched according to same grade level and teacher; similar math achievement, grade average, and pretest score; equivalent previous math course; and similar age. Students were matched according to their current classroom teacher so that the differences between instruction are not dependent upon class-wide teacher-student relationship.

## Rationale

Despite the emphasis on increasing standards, researchers have only begun to examine how mathematics curricula and pedagogy can be designed to improve the achievement of secondary students who exhibit learning disabilities. Typically, mathematics studies focus on elementary and middle school students without disabilities rather than secondary school students with learning disabilities. The need of students who are having difficulty with current pedagogical practices, namely, students with

learning disabilities, may be addressed with a new instructional program specific to algebra. Research aimed at providing answers to curricular challenges not only benefits students with learning disabilities but also teachers who are continuously faced with the challenges of teaching abstract concepts in a manner that students may comprehend. Students need to know not only how to solve problems but also why they are using certain techniques.

The purpose of this literature review was to analyze the differences in student learning from arithmetic to algebra and review effective practices for students with learning disabilities. To determine how to intervene in algebra for students with learning disabilities, the theoretical perspective of exogenous constructivism that is often associated with effective practices was identified as well as how the CRA mathematical curriculum sequence emerges from exogenous constructivism. The components of instruction needed within CRA were also described as well as the limitations with current practiced models.

## Definition of Terms

Concrete. The concrete step of the CRA sequence of instruction refers to use of real life objects (Underhill, Uprichard, & Heddens, 1980) to solve the algebra equations:

> A concrete learning experience is one that helps the learner relate manipulative processes and computational processes. The learner focuses on both learner-manipulated objects and the symbolic processes which accompany, correlate to, and describe the manipulations. (p. 28)

Concrete instruction assists students by teaching them stepwise procedures to solve equations. This manipulation allows students to see how variables can by multiplied and

divided and how numbers can be added and subtracted to change location in regard to the equal sign.

Representational. Using concrete steps only reduces the iconic symbol difficulty and is different than working with numeric experiences for students solving equations. The representational stage is the link between learning math hands-on and computing math equations on paper. The representational stage is a combination of the semi-concrete and semi-abstract levels (Underhill et al., 1980) due to the materials used at the concrete stage. The semi-concrete side of the representational stage uses pictures of the concrete objects for students to manipulate on paper. The semi-abstract level of the representational stage uses tallies to represent numbers from the abstract stage of CRA.

Abstract. When the student has learned the concrete and representational stages of the math concept, thus showing iconic relationships, then the student is ready for abstract work. The abstract stage does not allow students to use hands-on manipulatives nor visual stimuli to support their stepwise processes (Underhill et al., 1980). Abstract symbols used are letters, plus and minus signs, equal signs, and numerals.

Probe. Probes are supplementary assessment tools to learn about the student's math skill acquisition in a specific area.

Coefficient. A coefficient is the number of the variables represented by the numeral multiplied to or divided by the variable.

Variable. In simple equations a variable is known as an unknown quantity represented by a symbol in a math equation. In more complex equations, multiple variables exist in equations. In such equations the value of each variable changes with the value of the other variable. Typically, letters represents variables.

Learning disabilities in math. The student is recognized by the county as having a severe discrepancy between math ability and achievement. For this study, a learning disability was established as one and one-half standard deviations between ability and achievement and at least one math goal was listed on their individualized education plan.

At-risk for difficulties in algebra. Poverty is a strongly linked risk factor to disabilities Thus, students who reside in low socioeconomic status homes have increased risk for difficulties in academics (Prater, Sileo, & Black, 2000; Signer, Beasley, & Bauer, 1997), specifically math achievement (Schullo & Alperson, 1998). Linked factors include lower grade achievement, lower achievement scores and lower self-concept. For this study, the at-risk group included students with free or reduced lunch who performed at or below average (stanine of 5 or less) on the math section of the Florida Comprehensive Achievement Test (FCAT).

Direct instruction. Direct instruction applies explicit instruction, mastery learning, fading teacher involvement, examples and modeling, reviewing prior knowledge, and teacher-led instruction and correction (Maccini & Gagnon, 2000).

Transformations. An algebraic expression involving different variables and different coefficients of variables that need to be factored and reduced to its most basic components.

Arithmetic. Arithmetic is sequentially taught mathematics involving basic facts and other concrete oriented mathematics. This instruction is meant to help students understand the use of numerals as they represent numbers.

Algebra. Algebra introduces more abstract concepts where students solve for unknown quantities and chart how numerals in equations vary upon each other.

# CHAPTER 2
# REVIEW OF RELATED LITERATURE

This chapter covers the background information to the study. The difficulties typical to arithmetic and algebra are examined as is the connection between the two broad areas of mathematics. Strategies that have been established as effective for students with disabilities and how these strategies may be used to help students in algebra are discussed. These strategies are combined theoretically and practically to form a new model for teaching algebra. This model and the current study that develops from this model are introduced.

Three separate searches were conducted using the ERIC database. In reviewing literature on math graduation requirements, the search descriptors used were "secondary," "mathematics," "graduation," and "requirements." Using the search descriptors "teaching methods" and "mathematics" the researcher located literature in effective teaching methods. Finally, "learning disabilities" and "mathematics" were used to locate information on mathematics interventions and background for students with learning disabilities. From the available literature, four criteria were used for the inclusion of the reference:

- Subjects and settings were explicitly stated.
- All experiments were described thoroughly so that the procedures could be replicated.
- Interpretations and conclusions were consistent with the results and experimental design.

- The reference was published in the last 10 years (1989 – 1999), unless the contribution made by the reference was monumental to this research.

To find the current literature on algebra curriculum, the descriptors included were "algebra," "curriculum," and "research." First, the empirical investigations were reviewed, followed by professional literature that examined and expanded the area of secondary school mathematics. From the literature emerged a flowchart (see Figure 1) of the difficulties that students have in algebra and their relationship to common difficulties in arithmetic. The compounded difficulties that students with learning disabilities experience were also examined. Additionally, possible interventions mixed with methods of effective instruction transpired that may provide students with learning disabilities a more successful experience with algebra.

## Arithmetic to Algebra Gap

One of the hottest topics over the past couple decades has been whether or not algebra exists on a continuum with arithmetic. Typically, students take their first course in algebra following successful completion of advanced arithmetic with some introduction of unknowns. Problems with understanding and instruction between arithmetic and algebra have been challenged and upheld by different researchers (Herscovics & Linchevski, 1994; Kieran, 1992; Linchevski, 1995). Problems with understanding and instruction have been termed the arithmetic to algebra gap (Herscovics & Linchevski, 1994). Table 1 lists some major work that describes research behind the arithmetic to algebra gap.

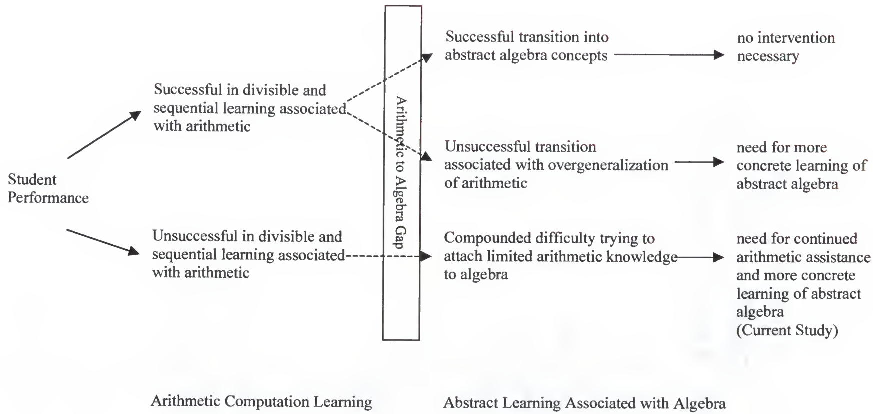


Figure 1. Flowchart of Algebraic Needs for Students Who Experience Difficulty in Math

Algebra as a Continuation of Arithmetic

Many researchers believe that to help students learn algebra it is best to start with their current knowledge on arithmetic (Herscovics & Linchevski; Kieran, 1992; Lee & Wheeler, 1989; Linchevski, 1995; Peck & Jencks, 1988). Using case studies, Herscovics & Kieran (1980) found that seventh- and eighth-grade students try to relate conceptual understandings of algebra back to arithmetical understanding. The attempt to connect arithmetic to algebra may cause confusion for students because of the lack of smooth transition from arithmetic to algebra, leaving major conceptual gaps. In an attempt to discover where the possible gap may exist between arithmetic and algebra, Herscovics and Linchevski (1994) interviewed 22 seventh-grade students and observed student think-aloud sessions to examine cognitive processes. They found evidence that a cognitive gap exists between arithmetic and algebra. Teachers may help bridge the gap through altering the pace of instruction and scaffolding from frames for unknowns (i.e., empty boxes as unknowns), such as $6 * \square = 18$, to $6Y = 18$. They also found that problems in the traditional left side of an equals sign should be presented before being switched to the right side, such $37 - N = 30$ before $30 = 37 - N$.

Kieran (1991) also stressed the use of students constructing their own meanings in algebra by teaching through frames first (empty boxes) and then variables. Kieran showed how a student's first use of frames starts by trial and error until the student discovers the answer to the empty box. Phillipp and Schappelle (1999) believed that while at times algebra involves meaningful manipulation of symbols, it also represents meaningless representations of symbols. That is, algebra involves manipulations of

Table 1

Summary Table for Arithmetic to Algebra Gap Research

| Author / Date | Participants | Methodology | Results | Weaknesses |
|---|---|---|---|---|
| Herscovics & Kieran (1980) | 7th and 8th grade students with initial prealgebra concepts. | Similarities collected from case studies of think aloud interviews where students discuss procedures to solve equations. | Students try to relate new algebra concepts to arithmetic background. | Poorly defined subjects. No work with why some students are successful transitioning. |
| Herscovics & Linchevski (1994) | 22 7th grade prealgebra students. | Think aloud interviews. | Cognitive gap exists between arithmetic and algebra. Language barriers. | No standardized instrument for observations of students. |
| Lee & Wheeler (1989) | 268 10th grade prealgebra and algebra students. | Comparison on test-interview data on students. | Algebra students confuse arithmetic because of notation. | No interrater agreement discussed. |
| Stacey & MacGregor (1997) | Over 1000 algebra students ages 11-15 in 24 Australian high schools. | Teacher interviews and observations with students. | Prior knowledge affects algebra as students confuse referents and use of variables. | Survey technique not discussed. |
| MacGregor & Stacey (1998) | 268 algebra students ages 14-16. | Test with follow-up interviews. | Understanding a problem and answering algebraic expressions are different tasks. | No standards in interview process discussed. |
| Filloy & Rojano (1989) | (not defined) | Think aloud interviews. | Arithmetic may interfere with algebraic thinking. | Poorly defined study. Not written to reveal scientific method. |
| Demby (1997) | 108 7th and 8th grade prealgebra students in Poland. | Test- interview data on students solve algebraic problems (equations). | $7^{th}$ grade students need more concrete lessons while $8^{th}$ grade students may handle more abstract notions. | Little description of hands-on or concrete work described. |
| Geary, Hoard, & Hamson (1999) | 55 at-risk 1st grade students v 35 normally performing peers. | One way ANOVAs for most categories, two way ANOVAs for others and multiple regression for identifying math achievement variables. | Students with identified math difficulties have weaknesses in counting, memory retrieval, and number production. | Inflated p values in fishing statistics. No use of Bonferroni/ Dunn. |
| Cawley & Miller (1989) | 220 students with learning disabilities ages 8-17. | Analysis of mean scores from WISC-R and WJ-R across age levels supported by strong alpha coefficients. | Students with LD perform far behind peers gaining less than 1 grade year's growth for each academic year. | Not longitudinal and uses only 2 LA school districts. Strong statistically. |

Table 1—continued.

| Author/Date | Subjects | Methodology | Results | Weakness |
|---|---|---|---|---|
| Algozzine, O'Shea, Crews, & Stoddard (1987) | 1098 10th grade FL students with learning disabilities. | Comparison of SSAT-II Math Skills Test scores. | LD had lower basic facts and scored significantly lower than peers. Low knowledge harms employability. | FL only. Recency of study requires new interpretation. LD grouped in all academic areas. |
| Cawley, Parmar, Yan, & Miller (1996) | 421 students ages 9-14 - 155 students with disabilities and 266 without. | Analysis of variance across items and correlations between subsets of questions. | LD grow in leaps and valleys with distinct gaps of knowledge. LD show more errors with algorithms. | Strong research but minimal discussion of different subjects such as algebra or prealgebra. |

equations that have little or nothing to do with the purpose of the equation, but a student only manipulates equations after he or she understands what the symbols mean. This disconnection leads to confusion for students trying to comprehend equations and manipulate equations to develop a solution. Phillip and Schappelle demonstrated this by manipulating an algebra equation with fractions using arbitrary symbols obviously different than original numerals to obtain a new, but equal, equation. Using arbitrary symbols shows that algebra equations can be manipulated with little regard to the numerals in the original equation.

Lee and Wheeler (1989) also challenged algebra as growing out of arithmetic. In test-interview data from 268 high school students enrolled in algebra or approaching algebra, they found conceptual understandings to be completely different between the groups. Students may confuse arithmetic procedures due to algebraic notation. Ludholz (1990) proposed that the transition to algebra from arithmetic is particularly difficult for some students. Ten percent of the student population enters algebra by the eighth grade, while 65% enter algebra in the ninth grade. Difficulty starting algebra may exist for the other 25% who either attempt algebra after sitting out a year from any math course or coming from remedial math programs.

The design of word problems with unknowns also evidences possibilities of a gap between arithmetic and algebra (MacGregor & Stacey, 1998). No longer are equations written so that the answer is solved at the right side of the equation. Algebraic equations involve variables situated in the middle of equations or the right side of the equals sign. A common problem students have in representing equations involves referents to represent variables, which are often based upon key words, such as h = height

and p = principals (Kinzel, 1999; Stacey & MacGregor, 1997). Using referents may confuse students when the algebraic concepts become more complicated and the number of variables increases. When given abstract equations, students who use referents connected to key words try to attach meaning to the symbols of the equations. For example, $X + Y = 10$, means the Y may stand for yellow instead of an unknown number. A study of eighth- through tenth-grade students learning algebra demonstrated that even advanced students have difficulty accepting algebraic notation, such as $(X + 5) / 2$, as an answer to a complex problem (Booth, 1988). Referents from notation are very difficult concepts for students because interpreting notation involves a three-way relationship among the person, the notation itself, and the identified referent (Kinzel, 1999).

In a collection of studies of over 1,000 Australian algebra students ages 11-15, Stacey and McGregor (1997) found that students have problems learning algebraic notation because: (a) they make intuitive assumptions about the unfamiliar system, (b) they make incorrect analogies between the symbol system and everyday life, (c) the variables interfere with learning new material, and (d) poorly designed teaching materials affect students' learning of notation. The latter problem of poorly designed materials may also account for the other three problems. That is, effective curriculum can answer the difficulties of implicit assumptions, analogies, and variables.

Filloy and Rojano (1989) learned from structured "think-aloud" interviews that arithmetic experiences possibly interfere with algebraic conceptual thought. Students perform differently between arithmetical algebra and algebra equations. They distinguish between arithmetical algebra equations (dual variables on only one side of the equation such as $6Y + 7X = 32$) and true algebra equations (equations with at least two variables

on each side of the equation such as 6M + 1N = 55 – 7N * M). Herscovics and Kieran (1980) found similar differences in case study analyses of seventh and eighth graders who were trying to solve algebra problems at the equals sign when variables were still present on both sides (X + 4 = Y) instead of manipulating the equation first.

A learning gap exists between arithmetic learning and algebra performance even though there appears to be some flow from arithmetic to algebra. When a student is unsuccessful in arithmetic, that student will have difficulty in algebra even if the student understands the abstractness of algebra. For example, a student may understand how the equations 3x + 4y = 12 and 5y – 3x = 26 meet at an intersection of two lines, but, the same student cannot combine the equations numerically if he or she cannot compute the arithmetic. On the other hand, success in arithmetic does not guarantee success in algebra. Using the same example, if the student does not know why the two equations come to a single answer, it is unlikely the student will realize that he or she should or could compute them together. Such possible difficulties from arithmetic to algebra show how arithmetic is a necessary but not sufficient condition to solving algebra.

Bridging the Gap

Although many researchers believe the gap from arithmetic to algebra requires implicit understanding where students develop their own connections and reasons for connections, other researchers have found curricular ways to bridge the gap. McConnell and Bhattacharya (1999) argued that students often lose sight of the original problem when they approach a problem algebraically. They stated that the logic of arithmetic helps students avoid confusion with typical abstract algebraic equations and even helps set up equations for algebra. McConnell and Bhattacharya stated that students need to be

shown explicitly the connection from arithmetic to algebraic equations to "help demystify the process for them and help see algebraic solutions in a new light" (p. 495).

Demby (1997) studied 108 seventh- and eighth-grade students in Poland as they prepared for algebra instruction in the following year. Through test-interview data, Demby found that seventh-grade students used more concrete representations and showed less preparation than eighth-grade students do for algebra. Demby warned against the normal algorithm approach to teaching algebra when he stated, "Transmitting algebraic rules by the teacher; memorizing them by students; and practicing them in a mechanical way is worthless" (p. 68). Demby suggested that the teacher should prepare diverse tasks of similar concepts to help scaffold new concepts from prior knowledge.

Regarding a student's use of algebra referents, Kieran (1992) believed the best method to solve the trouble students have with the concept of algebraic notation is to teach math symbols as a separate language. In a similar direction, Kinzel (1999) noted that teachers need to teach explicitly how to identify and label variables in an expression. The NCTM (1989) addressed this issue by stressing the proactive use of letters in Grades 5 through 8 to prepare students for the concept of variables representing numbers in equations.

## Preparing Prior Knowledge

Teachers can gradually transition students into algebra (Carnine, 1997) by introducing algebra slowly throughout grade levels (Meyer, 1999; Thompson & Davis, 1998; Zawojewski, Robinson, & Hoover, 1999). The concerns with the gradual introduction of algebra involves the spiraling instruction, having students repeatedly reintroduced to the same material, which has been known to confuse students (Carnine,

1997). The material may be forgotten by the time the next introduction is made. If proper instruction is maintained, it is cumbersome to repeat the purpose of a concept several times to students while only once bringing the concept to possible closure. Chandler and Brosnan (1995) also found that algebraic concepts were not introduced until the fifth grade when algebra is introduced as only 1% of their textbook.

While Filloy and Rojano (1989) recognized the difference between how a student addresses arithmetical expressions and algebraic expressions, they provided no recommendation as how to teach these equations properly. Herscovics and Kieran (1980) believed the best method for addressing the confusion between arithmetic and algebra is through explicit instruction of how to solve abstract problems. Phillipp and Schappelle (1999) offered the suggestion of a constructivist dialectical approach using relevant story vignettes for students to understand symbol notation and meaning. They believed the notation should emerge from the students, and the teacher should monitor their understanding through the vignette.

If a gap of instruction and understanding exists between arithmetic and algebra, then curriculum needs to be designed to ensure algebra skills are understood. For algebra to be understood, teachers need to instruct students to apply algebra in ways relevant to the student, prepare students from step to step by activating schema, use explicit instruction to teach basic algebra skills when learning is new or difficult, and teach concepts to completion and mastery. As indicated by the few attempts to combine these techniques, it is difficult to produce curriculum that works for every student. Proper curriculum becomes increasingly difficult to develop and deliver for students with

learning disabilities. Students with disabilities have often repeatedly failed academically because of their poor response to curriculum.

Potential for Arithmetic Difficulties for Students with Learning Disabilities

Geary, Hoard, and Hamson (1999) found that first-grade students with disabilities in mathematics already show difficulties with counting knowledge, number naming and writing, and memory retrieval as compared to their nondisabled peers. Geary and his colleagues argued that these early errors in basic math may affect their future mathematics learning. The authors did warn, however, that little can be assumed about the effect on algebra and geometrical understanding for students with disabilities because there is too little known about normal growth in algebra learning.

Gersten and Chard (1999) labeled the problems listed by Geary and his colleagues as difficulty with number sense. While the students may memorize some of the facts and be able to recite them in isolated settings, the students have difficulty working backwards due to their conceptual confusion. For example, a student may have memorized that 3+5=8, but the same student may have to repeat calculation processes when he or she sees 8-5 = ___. This lack of knowledge and ability to relate the first number fact to the second subtraction fact show that the student does not have number sense. Additionally, the student may also not understand the use of subtraction and addition. While knowledge of beginning math instruction is valuable for looking into reasons for difficulty in higher level mathematics, others have investigated mathematics growth later in the arithmetic curricula to find differences between skill acquisitions of students with disabilities as compared to their nondisabled peers. The more we understand about

acquisition differences between students with and without disabilities in arithmetic, the more we may learn about the problems students bring with them into algebra.

In a review of standardized mathematical performance of 220 students with learning disabilities in two Louisiana school districts, Cawley and Miller (1989) found evidence of developmental patterns across ages. Eleven-year-old students with learning disabilities were working from a 2.2 grade equivalency. Twelve-year-old students operated with 2.6 grade equivalency. Thirteen-year-olds worked with a 3.6 grade equivalency, and 14-year-olds averaged a 4.3 grade equivalency. While these scores do not measure the steady growth of the Louisiana sample, they are representative of scores from potential growth patterns of that county. These less than satisfactory growth patterns support the work of Algozzine, O'Shea, Crews, and Stoddard (1987). Algozzine et al. found in their analysis of a 1,098 10th-grade student sample that students with learning disabilities in Florida performed lower in basic math skills (e.g., decimals, whole number operations) than nondisabled peers.

Cawley, Parmar, Yan, and Miller (1996) supported the knowledge learned from these earlier studies in their comparison of 155 students with disabilities ages 9-14 to 266 normally achieving students. They found students with learning disabilities learn in "leaps" and "valleys." compared to students with normal achievement who learn math in a steady age-growth pattern. They also found that students with learning disabilities had more difficulty with algorithms and more computational errors than students without disabilities.

Much of this research showed what were potential problems for students with learning disabilities. Students with learning disabilities perform lower than their

nondisabled peers on computational and algorithmic exercises, possibly stemming from poor number sense. While these problems begin in primary grades, the evidence of an arithmetic to algebra gap presents the strong possibility that the problem will exacerbate students with learning problems' difficulties with higher conceptual mathematics. Therefore, to address difficulties with upper level mathematics, it is essential to consider how to address basic arithmetic simultaneously.

## Effective Math Instructional Strategies for Students with Learning Difficulties

While discovering components of effective algebra instruction are difficult, addressing the needs of students with learning disabilities in math classes is even more challenging. Aside from algebra, though, research in mathematics instruction for students with learning disabilities shows similar discoveries to the needs for bridging the arithmetic to algebra gap. Such ideas for helping students with learning disabilities are providing explicit instruction, cueing prior knowledge, and teaching through relevant activities using an exogenous constructivist instructional method. Table 2 lists some major research that necessitates different math strategies that help students with learning difficulties.

### Explicit Instruction

Jones, Wilson, and Bhojwani (1997) outlined methods for explicit curriculum design shown effective for teaching students with learning disabilities. The authors concluded that poor achievement of secondary students with learning disabilities in math is due not only to the students' lack of prior knowledge but also low expectations and poor instruction. Maccini and Hughes (2000) outlined a program to help students with

Table 2

Summary Table for Math Strategies Research with Students with Learning Problems

| Author / Date | Subjects | Methodology | Results | Weaknesses |
|---|---|---|---|---|
| Hartnett & Gelman (1998) | Two studies- 201 students total ages 5-7. | Case studies and structured interviews with teachers. | Implicit instruction only works if ideas flow smoothly and are easy for students to understand. | Neither study is well defined. While the ages do not match this research the points continue. |
| Hanrahan (1998) | 11th grade biology students. | Qualitative participant observation-interviews with a written response survey. | Students' autonomy builds motivation, but learning may still suffer when students are motivated. | Not a math research project as math is not as clearly lab oriented. |
| Fleer (1992) | K-3rd graders. | Structured interviews on understandings of scientific phenomena. | Teacher is critical to students understanding. Scaffolding student understanding starts with modeling and continues with student application. | Different age and subject to this research. Subjects' disabilities are not well-defined. |
| Dole (1990) | 6 students ages 12-13. | Use of hands-on manipulatives with repetition exercises. | Repetition and mix of hands-on curricula is more effective than hands-on alone in subtraction problems | Prior knowledge not completely accounted for in remediation study. |
| Funkhouser (1995) | 12 case studies of students with learning disabilities in K-1 classrooms. | Developing number sense with 0-5 integers. | When students are taught about a number first they can successfully compute simple equations using that number. | Case study approach with severe maturation validity issues. |
| Hutchinson (1993) | 12 adolescents with learning disabilities working on algebra word problems. | Single subject-multiple baseline followed on self-questioning techniques. | Metacognition is needed to think about the algebra process as it relates to word problems. | Representation of word problems and solving for word problems are different conceptually. |
| Zawaiza & Gerber (1993) | 60 community college students- 38 with disabilities and 22 without. | Measured three different groups across labels on the affect of concrete materials. | Concrete objects even out performance between students with disabilities and their non-disabled peers. | College students and secondary students are likely to handle abstractions differently. |
| Maccini & Hughes (2000) | 6 students with learning disabilities in high school algebra. | Single subject-multiple baseline. | CRA works for solving word problems. | Representation of word problems and solving for word problems are different conceptually. Model appears to be unable to help solve more complex problems. |

learning disabilities represent word problems based on direct instruction that includes a high degree of teacher-directed explicit instruction. In their single-subject multiple baseline study, they found that explicit instruction using manipulatives improves a student's ability to set up algebraic equations from problem-solving passages. Jones et al. (1997) urged teachers to connect concepts and skills, organize ideas in a hierarchy, and explicitly model and teach skills.

Hartnet and Gelman (1998) warned against implicit instruction in mathematics. In case studies with kindergartners and first and second graders, Hartnet and Gelman found that students learning implicitly works only if the ideas are smooth flowing, such as addition to subtraction and addition to multiplication. When the ideas do not flow easily, however, teachers need to provide structures for student understanding and concept development. Although they had no evidence to generalize beyond K-2, Hartnet and Gelman expected difficulties with lesson sequence to be persistent throughout grades. A suggestion from Fleer (1992) is to cue prior knowledge and scaffold understanding through interactive teaching. Fleer noted that the teacher is the critical component in building a student's understanding from manipulation of concrete (i.e., physical) objects to higher order thinking.

## Cueing Prior Knowledge through Scaffolding

Teachers can help students build knowledge to attain higher order thinking. They can do this by cueing students' prior knowledge to prepare them for the next concept. While Bybee and Sund (1982) stated that teachers should present material at or above a student's level, Fleer (1992) found that students learn more efficiently when instruction and discussion are at a student's level and not ahead of it. The concern is that if we are

constantly pulling students along, we may lose sight of their understanding and lose students before they get to mastery. Not achieving mastery is a common mistake (Deshler & Schumaker, 1993). The teacher begins to teach solely through lecture to speed instruction. While lessons designed solely around lecture are very popular, they are not considered in line with effective teaching practices (Garet & Mills, 1995) and have been shown to be detrimental to motivation (Hanrahan, 1998).

Developing Motivation through Relevancy

Hanrahan (1998) interviewed, surveyed, and observed 11th-grade biology students and determined that the degree of student autonomy affected motivation. The more choices students have and the more opportunity to provide input in classroom activities, the easier students can perceive relevance and thus increase motivation (Deci & Ryan, 1992; Hanrahan, 1998). Moses (1995) pointed to curriculum as a source of motivation. For example, he emphasized the power of students learning algebra because it brings them access to mathematical literacy and economic empowerment. Additionally, knowledge of algebra allows students to develop abstract notions that may assist them in future work after school. It is difficult for all students to understand the need for such abstractness or economic empowerment unless teachers explicitly teach these aspects of algebra in a manner that appears relevant to students. While motivation alone through relevancy is not sufficient to teaching algebra, it does provide a purpose for students to base their efforts. After teachers establish motivation, they must be able to implement effective practices.

Although there appears to be a connection between what practices exist for students with learning disabilities and how to bridge the arithmetic to algebra gap,

identifying effective curriculum for students with learning disabilities in algebra is difficult. The connection exists at explicit instruction, preparing prior knowledge, and teaching through relevancy. As Maccini and Gagnon (2000) pointed out, this connection also exists within the theoretical perspective called constructivism. To understand possible curriculum emanating from constructivism, it is important to examine constructivism itself.

## Constructivism

Constructivism's impact on education over the past few decades has affected students from kindergarten through graduate school. Kelly (1955), one of the fathers of constructivism, discussed constructive alternativism in which a student constructs his or her own knowledge rather than passively receiving it. Kelly proposed that we as people can change or replace our present interpretation of events. This implies that our behavior is never completely determined as we can reinterpret our experiences through metacognition and reflection. Kelly further explained that we personalize our judgments, and new elements can permeate into our personal construct framework.

Biehler and Snowman (1990) described constructivism as a balance between assimilation and accommodation in Piaget's formal operational stage of development. Assimilation is the interpretation of an experience so it fits into an existing scheme or thought. Accommodation is a change or development of a scheme to incorporate a new experience. Piaget (1963), another founder of constructivism, stated that construction of knowledge is aided by assimilation that leads to spontaneous constructions of knowledge. Such spontaneous constructivism accelerates development which aids not only

acquisition of knowledge but also retention and motivation. Some math work that explores constructivism is described in Table 3.

Table 3

Summary Table for Constructivistic Math Research

| Author / Date | Subjects | Methodology | Results | Weaknesses |
|---|---|---|---|---|
| Baird, Fensham, Gunstone, Penna, & White (1991) | 200 6th grade students and over 2000 7th-11th grade Australian students. | Meta-analyses of 4 different studies. | Most important piece to constructivistic instruction is reflection and meta-cognition. | Not specifically on students with disabilities and how to apply knowledge to students with LD. |
| Mercer, Jordan, & Miller (1994) | 14 instructional programs. | Qualitative breakdown of instructional programs through a checklist of instructional components. | Confusion exists over the term constructivism and how constructivism principles are applied. | No data on the success of each individual program are compared to the components listed as constructivistic. |
| Montague, Bos, & Doucette (1991) | 60 8th grade math students with learning disabilities who were low, average and high achieving. | Survey percentages and measures of variability. | Students with LD have difficulty applying mastered knowledge despite achievement level. | No clear conclusions on how to apply math, specifically algebra. |
| Batanero, Navarro-Pelayo, & Godino (1997) | 720 14 to 15 year olds working with prealgebra word problems. | Analysis of variance across instructional style. | Implicit instruction leads to misinterpretations of word problems and confusion of definitions. | More needed on applying principles of word problem representation to solving algebraic equations. |

Practical Constructivism

While Piaget and Kelly outlined the basic principles behind a person's use of constructivism, educators experimented on how to use constructivism's developmental concepts with curriculum. Hannifin, Hannifin, Land, and Oliver (1997) defined a constructivist implementation as follows: "Objects and events have no absolute meaning; rather, the individual interprets each and constructs meaning based on individual experience and evolved beliefs" (p.109). Hannifin and his colleagues also separated

constructivism from instructionism, where students' must recognize new information and integrate new knowledge with existing knowledge to learn. Instructionism is providing new information in a direct learning situation while constructivism is when students provide their own information through experiences. While this definition of instructionism revolves around Piaget's assimilation, integrating new knowledge with existing knowledge, constructivism allows both assimilation and accommodation.

The current definition of constructivism continues the traditions of Kelly and Piaget, but it has also expanded. The use of constructivistic principles in schools has revealed that constructivism has developed into many different levels. Hannifin and his colleagues described these levels as situated cognition, social constructivism, and pure constructivism. Situated cognition is the use of story vignettes to determine relevance; social constructivism is working with problems through collaboration with experts and novices; and pure constructivism is a student taking an individual project from start to finish. Moshman (1982) also proposed different levels of constructivism. Using Piaget's definition of linking prior knowledge to new knowledge, Moshman postulated a three-step continuum of constructivism from endogenous, to dialectical, to exogenous. Endogenous constructivism is when the transfer of knowledge from the world to the student occurs within the student. Dialectical constructivism means the student learns through a collaborative effort with the subject being studied. Exogenous constructivism occurs through explicit direction by the teacher. Hannifin and his colleagues and Moshman made it clear that constructivist principles can be followed under teacher and expert supervision. While students construct their own understandings, the teacher as

well as the activities in the classroom can aid in a student's construction. The amount of assistance and aid decreases as each continuum approaches the pure constructivism stage.

Constructivism components. Constructivism is often confused as only student-directed and initiated learning. On the contrary, constructivism can contain many methods for effective teaching. Several components to instruction and curriculum emerged from literature on constructivism. The components are scaffolding, mastery teaching, purposeful teaching, and proper assessment.

Scaffolding and explicit instruction. Bruner (1983) introduced the idea of scaffolding to describe how teachers can transition a student from one stage of learning to the next. The positions of a student's development (Piaget, 1963) go through three stages—from transmission, to maturation, to construction. Transmission is a student filling information into an old scheme. Maturation is the developing of a student's conceptual knowledge, and the construction stage is when a student interacts actively to learn. Such scaffolding is through socially constructed learning. This dialectic humanistic approach to learning involves participation from both teacher and student. In line with the work of Hannifin et al. (1997) and Moshman (1982), the goal of scaffolding is to provide a challenging form of instruction that eventually will allow students to learn themselves (Hogan & Pressley, 1997).

Mastery teaching. Since the teacher is considered the critical component in a student's construction of knowledge through scaffolding (Bybee & Sund, 1982; Fleer, 1992), there have been guidelines as to how a teacher can aid a student's understanding using proven curriculum. Flavell (1985) noted that students often have difficulty with constructivist classrooms. Students have been shown to have difficulty learning initially,

what Flavell called mediation deficient, but as learning increases, academic tasks become automatic and spontaneous (Flavell, 1985). This initial difficulty was explained as possibly being the result of a student's poor use of assimilation instead of accommodation in skill acquisition. For example, instead of a student learning to solve for a variable on either side of the equal sign, the student may try to solve every algebra equation to the right of the equal sign. To help students adapt, teachers have been encouraged to move a student through the curriculum one task to the next only after the student has mastered the previous concepts (Simon, 1995). This mastery allows the student the opportunity to appropriately assimilate their learning or accommodate for new concepts.

Purposeful teaching and relevancy. A teacher can aid a student's construction of knowledge by introducing problems and conflicting situations through challenging vignettes within a student's capabilities (Baird et al., 1991; Bybee & Sund, 1982; Hannifin et al., 1997). Baird and his colleagues, in an analysis of four studies involving over 2,000 6th through 11th graders, found that the most important processes in a student's construction of knowledge are reflection and metacognition. Reflection and metacognition are aided by challenging students through purposeful inquiry about meaningful activities to the student. Purposeful inquiry gives a balance between a student's cognition and affect. That is, the student is not only required to think about the problem but also is motivated to continue to find an answer.

Assessment. The purpose of assessment is to determine the amount of students' learning specific to what is being tested. Hannifin et al. (1997) urged teachers to match teaching technique to assessment (e.g., constructivism with relevant tasks). They

continued by explaining how too often teaching is conducted in a constructivist nature while students are assessed according to their rote learning. For example, after a field trip to a wildlife preserve where students examine a stump full of microorganisms, the teacher should grade the students on their reflections of the micro-organisms rather than a multiple-choice exam. By matching lessons to assessment, a teacher can identify more accurately where the student is having difficulty with the concept rather than difficulty with the assessment itself.

Misconceptions about Constructivism

Although in this review the ideas of constructivism were shown to have many variations in different classrooms, many educational programs still claim only pure constructivism is the best way to teach a student. Mercer, Jordan, and Miller (1994) challenged the qualities of educational programs that are labeled as pure constructivistic. In an analysis of 14 instructional programs labeled as constructivistic by the publishers, Mercer and his colleagues collected 19 themes that were consistent in all the programs. They learned that 5 of the 19 common characteristics outlined teacher-directed instruction. Twelve of the 14 programs included modeling of the target strategy; 9 of the 14 stressed dialogue; 5 of the 14 used explicit instruction; and only 2 of the 14 programs used peer collaboration. These findings show that although many people may feel constructivism is implicit instruction, in practice constructivism can contain components of explicit instruction.

Explicit instruction in isolation, though, is not constructivism. Constructivism must attach student input and creation in a manner that makes conceptual knowledge relevant to the student. While a purpose of constructivism is to attach relevancy, students

may also have difficulty applying what they know to real world problems. Montague et al. (1991) found that, although students with learning disabilities may master strategy instruction, they had difficulty using this knowledge with word problems. Students with learning disabilities were unable to visualize the word problems they were working with and were also unable to restate word problems.

## Implicit Instruction

The expanse of constructivism ranges from implicit teaching and self-discovery to Moshman's (1982) exogenous constructivism using explicit instruction. Like Mercer and his colleagues (1994) discovered, many constructivist programs use explicit instruction because implicit instruction was shown to be troublesome to some students in mathematics. Batanero et al. (1997) noted several difficulties in teaching secondary math students implicitly. In a study of 720 14- and 15-year-olds on combinatorial capacity, Batanero and colleagues found implicit instruction led to (a) misinterpretation of the problem statement, (b) confusion of definitions and the type of objects used, (c) exclusion of important elements to solving the problems, and (d) incorrectly labeling and using math operations and referents.

Otte (1998) also questioned implicit constructivist principles through analysis of the work of Kant, Peirce, and Piaget. Otte stated constructivism works only if the interactions between the knower and the environment yield a continuity and generalizing tendency rather than merely being seen as singular events. Kant and Peirce believed a person's perceptions can be faulty and that Piaget's "reflective abstraction" is different from true empirical abstraction. Otte stated that if the student does not follow empirical research, the student may not develop proper ideas.

## Constructivist Math

Batanero et al. (1997) led us to understand how pure constructivism can be troublesome for a student in secondary mathematics but that constructivist teaching using the continuum of constructivism can be beneficial for students with learning disabilities. Some educators use constructivistic math models while maintaining principles of effective instruction for the purpose of helping students acquire and retain knowledge (Simon, 1995; Steffe & D'Ambrosio, 1995), but implementation of constructivism with math can be difficult. Due to their exposure to the subject, many teachers who are experts in algebra believe algebraic understanding is implicit (Blais, 1988; Eisner, 1982; Goodson-Espy, 1995). Because experts feel algebra is implicit, they teach algebra through a series of algorithms and thereby provide shallow background to conceptual understanding. Blais (1988) believed this lack of proper pedagogical practices has led to mathematical atrophy because students readily forget the algorithms since there was little or no deep understanding.

### Explicit Instruction and Cueing Prior Knowledge

To achieve deep understanding, Mercer, Lane, Jordan, Allsopp, and Eisele (1996) discussed the continuum of instructional choices by listing mathematical examples at each step. Explicit instruction includes, but is not limited to, teacher models, think-alouds, cues, and prompts. Implicit instruction involves linking prior knowledge, reflecting on learning, and using interactive discourse and discovery methods. Effective instruction is when explicit and implicit instruction is blended to account for various learning styles and student strengths.

Transfer of skills from lesson to conceptual understanding requires explicit training (Campione, Brown, & Bryant 1985; Paris & Oka, 1986; Pressley, Forrest-Pressley, Elliott-Faust, & Miller, 1985). When skills are not explicitly taught, teachers run the risk of students learning by trial and error. This not only creates time constraints in the classroom but also hinders motivation of the students. Mercer et al. (1996) explained the need for teachers to blend techniques to reach an entire class of students with different preferred learning styles and strengths. Meyer (1999) also challenged teachers to create a classroom open to several strategies to account for differences in students. Meyer challenged teachers to allow students to use simple models for solving complex math concepts until the students develops more complex and abstract thought.

## Relevancy and Motivation

Mercer, Jordan, and Miller (1996) tied Moshman's continuum of consructivism to blending explicit and implicit instruction while using class activities that involve "real world" problem solving. Explicit instruction needs to be used for tasks that are new, and implicit instruction needs to be used for authentic learning so that students can connect meaning and relevance to the learning task. For example, concrete lessons emphasize explicit instruction by focusing on the manipulation of the material that allows the student to attach personal meaning to the content, thus making the material authentic. Another way to attach meaning is through problem solving (Mercer et al., 1994). The NCTM (1989) emphasized problem solving as the most important feature to mathematics for future vocational reasons. Mercer et al. emphasized that, while explicit instruction is necessary, problem-solving strategies must still have authentic context. In other words, if $6y + 2y = 16$ is not pragmatic, then students are merely memorizing algorithms.

Use of CRA with Basic Math Facts

Carnine (1997) also emphasized the need for explicit instruction using big ideas and allowing time for students to practice and review. Big ideas are main concepts to be taught prior to skill lessons. He admitted there is little research on the effects of proper instruction or the continuum of instruction with secondary students, but he stated the principle of best practices emphasizes scaffolding transition of self-directed learning from explicit to implicit instruction. This scaffolding allows students to "ease into" complex strategies. One type of instruction that provides the opportunity for students to ease into instruction through big ideas followed by detail is the CRA sequence of instruction. Table 4 lists some major arithmetic research on CRA instruction for students with learning difficulties and disabilities.

Table 4

Summary Table for CRA Basic Facts Research

| Author and Date | Subjects | Methodology | Results | Weaknesses |
|---|---|---|---|---|
| Miller & Mercer (1992) | 8 teachers with 15 elementary students with LD, 18 at-risk, 4 ED and 3 SED. | ANOVA on pre,post and follow-up scores on place value and basic facts. | Validates CSA as a means for teaching students with difficulties basic arithmetic. | Arithmetic growth and algebra are different conceptually. |
| Miller &Mercer (1993) | Nine students ages 7-11 with math disabilities. | Case studies with follow-up abstract data. | Positive retention with CRA; skill acquisition obtained w/n CSA. | Individual based program only can be used for individualized instruction and no mention of use in algebra. |
| Harris, Miller, & Mercer (1995) | 112 2nd grade students (12 with LD and 1 with ED). | Multiple baseline across 3 classrooms using alternate forms of fluency. | Students with disabilities performed similar to peers except in word problems. | assessment not well defined and no comparison to other math programs. |
| Jordan, Miller, & Mercer (1999) | 125 students with and without learning disabilities. | Split plot ANOVA on growth and comparison in acquisition and retention. | CSA taught students performed slightly above textbook taught peers. | students at-risk benefited from the intervention but those with disabilities did not perform significantly better. |

The CRA sequence of instruction takes students from the concrete step using manipulatives to the representational step using pictures to the abstract step of numerals and mathematical symbols. The representational pictures relate directly to the manipulatives and set up the standards to solve equations with numerals. Tying representations to abstract and building accuracy with numerals is the abstract step to CRA. This CRA sequence of instruction has been successful with basic facts and appears to tie directly to algebra. Researchers have demonstrated how teaching math facts through the CRA sequence of instruction (i.e., concrete, representational, abstract) accomplishes this continuum (Mercer & Miller, 1992, 1993). Miller and Mercer (1993) found success when using CRA sequence of instruction for teaching basic facts. In case study analyses of their strategic math series in addition, subtraction, multiplication, and division, Miller and Mercer found that not only did students acquire the proper knowledge of the subject, they also retained their understanding over 3 months later. Harris et al. (1995) in a replication study of the CRA sequence of instruction for solving multiplication facts with second graders found that students with learning disabilities who learned through the CRA sequence of instruction performed as well as peers who learned through traditional instruction thus equalizing achievement. The one area in which they did not find equalized achievement was in solving word problems. However, solving word problems involves more than mathematical intervention for some students who may require additional instruction in reading comprehension.

While the CRA sequence of instruction has been shown to be beneficial for students with disabilities to learn basic math facts there has been little research with secondary level concepts. One study on CRA in advanced elementary concepts was by

Jordan et al. (1999) who found some positive growth for low socioeconomic students at risk for math difficulties. However, students with disabilities who were taught with a CRA-type instruction did not show significant improvements over peers taught abstractly. Signs of such growth with students at risk for disabilities leads to the notion that the CRA sequence of instruction has potential to help students with higher conceptual mathematics learning. Algebra is a higher order conceptual learning area that appears important to investigate with the CRA sequence of instruction.

## CRA Principles Used with Algebra

Miles and Forcht (1995) conducted a pilot study of the Cognitive Assault Strategy, a plan that uses a 1:1 mentor system to increase conceptual math understanding. They found that difficulty in mathematics begins at the abstract level, which, the authors say, becomes increasingly difficult in algebra and calculus. Most of the examples of CRA sequence have been with elementary age students learning basic facts. While no research-supported and published CRA algebra programs exist, there is evidential need for the development of such a program. The research listed in Table 5 describes different aspects of CRA instruction for students with difficulties and disabilities applied to algebra.

## Algebra through Concrete Manipulatives

In one of the few empirical studies of the arithmetic-to-algebra continuum, Goodson-Espy (1995) recorded 13 college students' understanding of linear inequalities. Goodson-Espy discovered that a student needs to obtain levels of abstraction (complex thought) in terms of processes or even structure of the equation to demonstrate mastery.

She reported that for a student to understand processes and structure, the student needs to be involved in meaningful problems so he or she can relate processes to objects. Without understanding the concrete representation, a student will be unable to understand the purpose of the concept being taught. While many students may be able to implicitly link abstract lessons to relevance, many other students need to be shown the importance of lessons. Concrete manipulatives can give students an opportunity to understand purpose.

Table 5

Summary Table for CRA Components with Algebra Research

| Author and Date | Subjects | Methodology | Results | Weaknesses |
|---|---|---|---|---|
| Goodson-Espy (1995) | 13 college algebra students with difficulties. | Think aloud interviews with students with abstract versus concrete instruction only. | To master algebra, students need to reach an abstract stage of understanding instead of concrete alone. | College algebra and introductory algebra are different conceptually. |
| Hutchinson (1993) | 12 adolescents with learning disabilities working on algebra word problems. | Single subject-multiple baseline followed on self-questioning techniques using representations. | Metacognition through representations is needed to think about the algebra process as it relates to word problems. | Representation of word problems and solving for word problems are different conceptually. |
| Zawaiza & Gerber (1993) | 60 community college students- 38 with disabilities and 22 without. | Measured three different groups across labels on the affect of concrete materials. | Concrete objects even out performance between students with disabilities and their non-disabled peers. | College students and secondary students are likely to handle abstractions differently. |
| Maccini & Hughes (2000) | 6 students with learning disabilities in high school algebra. | Single subject-multiple baseline. | CRA works for solving word problems. | Representation of word problems and solving for word problems are different conceptually and model appears to lose ability to help solve more complex problems. |

Algebra Using Representations

Peck and Jencks (1988) proposed the use of graphical techniques as a representation activity for expressing algebraic quadratic expressions, such as (x-2)(x+4). Meyer (1999) also emphasized the need for students to use graphical patterns that can be built upon for more complex understanding. The frames or empty boxes, described by Herscovics and Linchevski (1994) and Kieran (1991), are a form of representation for the variable, but there is a need to represent more than the variable alone. The interactions of the variable and number together need to be represented. There has yet to be a researched and published product that represents the coefficient, variable, and number properly.

Working with word problems, Hutchinson (1993) found that emphasis on metacognition was effective for classifying and representing complex algebraic word problems for 12 adolescents whom she trained in problem representation and problem solution. This agrees with the work of Montague et al. (1991) who claimed that students with learning problems have difficulty with word problems because they are unable to visualize or restate word problems.

Zawaiza and Gerber (1993) worked on the visual needs for students with difficulties in algebra word problems. They studied word problem strategies with 38 community college students with learning disabilities and 22 math competent peers. Students were split into three groups across math competence labels. One group had to verbally translate word problems; another had to diagram word problems; and the control was required to maintain high on-task attention while solving word problems. Zawaiza and Gerber found that students who work with diagrams of concrete objects

representations outperform peers who were merely on-task and those who only verbalized problems. This was consistent across all three groups. Zawaiza and Gerber concluded that instruction can be modified to help adult students with learning disabilities conceptually map word problems. In other words, after students examine and use concrete manipulatives, they will have the prerequisite ability to develop representational maps. This representational mapping can aid in understanding the problem.

Not all research on representations are positive. One researcher found students have difficulties with representations when the representations do not generalize to more complex equations (Cifarelli, 1993). After his work with 14 college students, Cifarelli concluded that traditional forms of representations need to be reconsidered. For example, if a model helps students solve equations such as $3X = 27$, the model should also be able to represent $3X^2 = 27$. While Cifarelli's research reported representations as a drawback to more complex learning, he was using algorithms instead of pictorial representations after learning from concrete manipulatives. It is logical to think that if a representation is unable to be generalized to more complex work, then we need to be careful about what representations are used and how they are implemented.

## Problems with CRA Use

The effects of CRA are generally positive, but there are problems that can occur if the CRA is not used in full and proper sequence or if the lessons are not delivered using effective instruction techniques identified earlier in this review. If a mathematics model does not follow the sequence of instruction which enables a student to apply the new knowledge to abstract numerals, then the students will not be able to experience the

benefits of the instruction. Likewise, if a model does not generalize to more complex equations, then the model does not accurately represent the math behind the model.

Lack of pictures that translate to abstract. Zawaiza and Gerber (1993) found that while diagramming word problems was highly valuable to success, students have more difficulty representing word problems than actually computing abstract equations. Students demonstrated this by writing inaccurate equations from diagrams meant to conceptualize a word problem. The authors studied community college students, and they did not generalize their findings to secondary education students. They also did not address the need for explicit instruction in teaching students how to create representations. It would have been difficult for Zawaiza and Gerber to do so since there are few published systems of representation styles that conceptualize algebraic expressions and fewer that have been researched. A representation system or set procedures for students to follow when choosing referents and making conceptual maps to conceptualize word problems needs to be developed.

Willcutt (1995) developed an algebra program using blocks for students to use as manipulatives. The program, *Cubes*, is a discovery-based algebra series for students to transition from concrete manipulatives to symbolic reasoning in mathematics. While Willcutt developed the instructions to develop representations, student development of pictorial representations is not emphasized in the lessons. Also, Willcutt did not report any quantitative or qualitative outcomes for students as a result of the lessons.

Babbitt and Miller (1996) reviewed the current hypermedia present in computer programs today that are capable of modeling cognitive and metacognitive processes for basic mathematics facts. These computer programs are easy to use for beginning

computer users and can offer the pictorial concepts needed so students can apply metacognitive analysis to solving word problems. Ease of use is important as so many students with learning disabilities in mathematics also have difficulty with reading. Babbitt and Miller also emphasize the importance of math applications in everyday life. Relevant math applications fall in line with the NCTM (1989) standards that advocate problem solving as the top priority in math education.

Lack of full sequence of CRA. Researchers have attempted teaching algebra starting with concrete manipulatives (Howard, Perry, & Conroy, 1995). The reason for many of these failures is the lack of connection associated with manipulative and abstract equations. For the CRA sequence to go through the entire instructional continuum, not only must concrete manipulatives be used, but also pictorial representations need to be developed that can be generalized across the many complexities of algebra. Representations are emphasized as a need for students to understand algebraic concepts, but the concern is that students try to generalize conceptual models when the conceptual models do not generalize (Zawaiza & Gerber, 1993).

Borensen (1997) developed a patented algebra model that incorporates both concrete and representational levels. While Borensen has provided neither qualitative nor quantitative data on his model to support claims of improving algebra understanding, the attempted use of CRA instruction is evident in his model. However, the concrete component demonstrated by teachers does not translate directly into student action. Students must use a mixed representational and concrete instructional means meant to match the teacher's hands-on demonstration. For example, the teacher presents an equation on a scale-like apparatus in front of class showing how the components balance

across the equal sign. The students then use similar component materials to the teacher on a piece of paper rather than the teacher's scale-like apparatus. While concern exists over the appropriateness of the materials and their equivalent match to algebra equations, the sequence of instruction is also incorrect. A proper representation system is needed that not only connects concrete objects to abstract equations but also can be used in a variety of equations from single variable to multivariate equations.

## An Empirically-based Algebra CRA Model

One model that has shown promise to setting up calculations was Maccini and Hughes's (2000) model for algebra. They found in a multiple baseline study using 6 high school students with learning disabilities that the students benefited from the mathematical model using single variable equations with addition, subtraction, division, and multiplication. These results expressed the possibilities for a concrete-representational-abstract model in algebra. However, the model did have weaknesses. Like Borenson's (1997) model, the concrete and representational pieces use colored blocks to represent variables. While this representation covers single variable inverse operations, it does not generalize to more complex equations where the coefficient is greater than one. For example, 2X with a coefficient of 2 and the variable/unknown X could be represented by 2 yellow blocks. This model may then be troublesome to students trying to use squared variables, such as $X^2$, for quadratic equations or parabolic graphing. Additionally, division of the coefficient, two yellow blocks, is unclear as to how numbers may divide out of each block. The model appears to eradicate the variable all together in division steps.

## Elementary versus Secondary School

Howard, Perry, and Lindsey (1996) surveyed 249 teachers with 10 or more years of experience in southwest Australia. They found that, while 62% of primary teachers use manipulatives, only 8% of secondary educators use manipulatives for mathematics. They also discovered that not only did the use of manipulatives vary, but also which manipulatives are used differs.

Howard, Perry, and Conroy's (1995) examination of the use of manipulatives with secondary and elementary school teachers in Australia shed light on the fact that not nearly as many secondary education teachers use manipulatives as do elementary teachers. Reasons why this dramatic difference exists may be that there have been far fewer sets of manipulatives developed commercially for secondary level courses. Also, some teachers may feel that manipulatives for secondary education do not easily translate into abstract proficiency with the same concept. The need for translation of manipulatives to numerals with secondary students has already shown its presence in the lack of success that some constructivist math programs have had when they do *not* use the full CRA sequence. Another possible reason why secondary teachers do not use manipulatives in their classrooms may be that students are hesitant about playing with toys that may be construed as "childlike." Clearly, it is understandable that a primary concern for adolescents is that they do nothing to embarrass themselves. Concrete manipulatives need to be age appropriate and relate easily to complex concepts.

Another reason why some teachers may not use concrete manipulatives is the time constraint on educators. Using concrete objects and representations while teaching each stage of CRA to mastery will require a great deal of time. Time may be a reason why

few attempts at adding manipulatives or pictures to algebra have been made. Algebra textbooks are usually dense with material and lessons, and experienced algebra teachers know they must move fast to complete the lessons designated by the set curriculum. Time appears to be an unsatisfactory reason for not providing proper instruction since the cost of speeding instruction is reduced student understanding of the concepts. While some school districts have changed algebra programs to 2-year tracks, such changes are rare.

## Summary

Students with learning disabilities experience difficulty acquiring and retaining algebraic concepts. These difficulties have emerged in national testing, state graduation exams, and classrooms across the world. Responsibility lies within researchers, curricularists, and teachers to help students with disabilities succeed in algebra.

The one major theme that has emerged from this review is the need for the CRA sequence of instruction to be applied to teaching algebra to students with disabilities. To apply CRA sequence to algebra, instruction needs to include relevant and explicit instruction. Relevance and explicit instruction are not only parts of the theoretical perspective of constructivism, they are also the elements that have been found successful in teaching mathematics to both students with and without learning disabilities. Students taught basic mathematics facts using the CRA sequence of instruction have demonstrated improvements in acquisition and retention of mathematics concepts (see, e.g., Mercer & Miller, 1992; Miller & Mercer, 1993).

For the CRA sequence of instruction in algebra to be effective, teachers must include manipulatives and representations that not only satisfy the objective of a single

lesson but also apply to the entire course curriculum. The problem that researchers have had when attempting to develop an algebra program applying CRA sequence is creating an R (representational) step of instruction that is generalizable to more complex algebraic equations. For example, a pictorial representation used for single variable equations needs to be expandable to multiple variable equations with fractions, exponents, and coefficients other than one.

The steps to the CRA sequence have not yet been validated to complex equations or group instruction with algebra. When researchers attempted to show causation for the CRA sequence to help students with learning disabilities in algebra, their findings focused on only a single principle of algebra. For example, a study on the CRA sequence may only present a representation step that links directly to a specific word problem rather than algebraic equations overall. A representational step needs to be developed that generalizes to broader use of arithmetic and complex algebra problems. Until generalizable concrete and representational steps are developed for teaching algebra, teachers are left with curricula that do not account for the arithmetic to algebra gap.

While theory indicates a hands-on and pictorial approach to be beneficial for students with disabilities in algebra there has been little research to confirm these beliefs. The current study compares a newly developed concrete to representational to abstract sequence of instruction model versus traditional abstract instruction on algebraic learning. Assessments were designed to compare the effectiveness of each instruction on acquisition and retention of knowledge.

# CHAPTER 3
# METHODS AND PROCEDURES

## Introduction

This study was designed to compare the effect of CRA sequence of instruction to traditional abstract instruction on acquisition and retention of transformational variable algebra equations. Chapter 3 outlines the research methodology, including a description of the hypotheses and a description of sampling procedures, subjects, and matched pairs. Additional sections of this chapter include details of the experimental design, implementation procedures, and experimental analysis.

## Hypotheses

A study of the effects of the CRA sequence of instruction on algebra acquisition and retention was conducted. Teachers trained in the CRA sequence of instruction used scripts and manipulatives to teach students single variable equations and transformation of algebraic expressions over a 4-week period. Tests of significance were run to discover what effects the CRA sequence of instruction had on algebra acquisition and retention for students struggling in mathematics? The following null hypotheses were run at the 0.05 significance level.

$Ho_a$: There will be no significant difference on measures of solving for a single variable in an algebra equation requiring transformations between students receiving CRA instruction and those who receive abstract instruction following instruction.

$Ho_b$: There will be no significant difference on measures of solving for a single variable in an algebra equation requiring transformations between students receiving CRA instruction and those who receive abstract instruction obtained 3 weeks after instruction has ceased.

## Methods

### Settings and Participants

The purpose of this section is to describe the instructional settings where the intervention took place and the subjects who participated in the study. Demographics are provided specifically to each classroom where the treatment was implemented. Procedures for assigning subjects to treatment and control are also provided.

School demographics. Four schools in a school district in west central Florida were selected for the study. Each school selected had a low overall income level as indicated by the percent of students who received free or reduced lunch. While 9 teachers signed up to participate in this project, 1 special education teacher in a self-contained setting dropped out due to personal reasons. Eight teachers remained and finished the project. Of the 8 teachers who carried the project through to completion, 4 taught mathematics for sixth graders and 4 team-taught two mathematics classes for seventh graders. Every class included students with and without disabilities. One sixth-grade teacher was certified in teaching students with learning disabilities, and only 1 other sixth-grade teacher had taken courses on students with learning disabilities. Both seventh-grade classes contained a math teacher and 1 teacher certified in teaching students with learning disabilities. The regular education math teacher took the lead in planning and implementing lessons in both team-taught classes. The special educators

monitored student behavior and answered all students' questions so as not to interfere with the instructional lesson.

Participants description. Secondary school students with and without learning disabilities in an urban county participated in this study. All students were included in the regular educational classroom for prealgebra and benefited from the guidance of both a regular education and a special education teacher outside of class. Descriptions of students include age, grade, previous math course, pretest score, FCAT achievement score, and whether the student was at-risk for future difficulties in math or had an identified learning disability in math.

A student considered to be at risk for algebra difficulty was one who performed at or below average on standardized math achievement scores and came from a home with low socioeconomic status (SES). It was important to take into account the SES and current performance level of students who may not have been labeled as learning disabled in mathematics. Students identified as at risk in this study scored a stanine of 5 or below in mathematics and problem solving on the Florida Comprehensive Achievement Test taken on March 6, 2000. Low standardized achievement scores are important to a student's growth in algebra. As previously stated, success in solving algebra equations is dependent on arithmetic knowledge. Thus, if a student has not performed well in arithmetic, then the student will likely not perform well in algebra. Semester grades, while used for matching students across the same teacher, were not used to identify at-risk students for advanced conceptual mathematics. Reasons for not using grade averages for identifying students was because teachers vary in their grading

procedures and weights for overall class grades. These procedures and weights were not accounted for prior to the implementation of this program's instruction.

Students identified as at risk came from homes that did not provide economic security. Students with low SES may have positive beliefs about education, but they perform below average in achievement (Mickelson, 1990). Moreover, Trusty (2000) found in a study of academic expectations that lower SES and lower middle-grade math achievement both significantly are related to expectations of high school students. Low SES is important to the growth of students in algebra acquisition because only certain students could ask for help at home. Since SES is related to education achievement (Mickelson, 1990; Sewell, Haller, & Portes, 1969), it may be assumed that lower SES groups may have not completed upper level mathematics courses and may not have performed upper level math equations for a number of years. Using county data, students who receive free or reduced cost lunch at school were considered to be from low SES homes. Students who fit both criteria--received free or reduced lunch and FCAT stanines of 5 or below and did not have an identified learning disability in math--were labeled as at risk for the purpose of this study.

Students identified as learning disabled had come through eligibility and had been identified through school services as those who needed additional support as evidenced by a one-and-one-half standard deviation difference in ability and achievement. Since math disabilities do not always, although often, coincide with reading disabilities, the students identified as disabled had math goals listed in their individualized education plan. If students only had reading disabilities or other disabilities, they were then excluded from the learning disabilities group for the purpose of this study.

Subjects were matched according to previous math course, Florida Comprehensive Achievement Test math score within one stanine, age within one year, same grade level, at-risk or disability label, semester grade average within two grade letters, pretest accuracy within one item, equivalent previous math course, and same teacher. Matching increased the power of the study by reducing the error in results due to previous learning, age, grade, disability label, and pretest. Each teacher who participated in the implementation of the study taught two equivalent algebra classes. In the first algebra class the CRA sequence of instruction was implemented, while in the second algebra class traditional abstract instruction was implemented. To account for differences between teachers, students were matched across teachers. To determine the effects of different instruction on different students, students were also matched across type of instruction. For example, in teacher X's first algebra class, there was a student who took basic sixth-grade basic math the previous year, is 12 years old, is now a seventh-grade student with a learning disability in mathematics, scored one answer correct on the pretest, and scored an FCAT stanine of 4. The match to this student came from teacher X's second algebra class. The matched student is also currently in the seventh grade who took sixth-grade basic math, is 12 years old, has a learning disability in math, scored no answers correct on the pretest, and has an FCAT stanine of 4. Descriptive group information is listed in Table 6, and specific student matches are listed in Table 7.

Since subjects were matched in pairs, some subjects had to be eliminated from data analysis. Subjects were eliminated if they missed a lesson and were unable to make up the lesson, if they missed an assessment and were unable to makeup the assessment, if they moved to a different class, or if parents or the student did not sign the permission

forms or refused participation. In such cases not only that student's data were eliminated but their matched partner's data as well. Additionally, data were not used when teachers did not deliver the instruction as specified by the treatment integrity assessment. Several students had to be eliminated from the data analysis because they had no county documentation as to their FCAT score, disability label, or economic status. According to teachers, grade differences were often due to homework completion.

By matching students, the power of the statistical analyses increases the likelihood the results may be attributed to the CRA sequence rather than confounds from age, grade, or previous learning. Additionally, using scripts and measuring treatment fidelity reduces the likelihood that differences between students' scores were due to the teacher's style or ability.

Table 6

Descriptive Information for Groups

| | Experimental | Comparison |
|---|---|---|
| Mathematics Ability | | |
| FCAT | M(SD) 4.088(1.311) | M(SD) 4.176(1.585) |
| | Mdn 4.00 | Mdn 4.00 |
| Pretest score | | |
| | M(SD) 0.12(0.41) | M(SD) 0.06(0.34) |
| | Mdn 0.00 | Mdn 0.00 |

Research instrumentation. To measure the acquisition and retention of knowledge on single variable equations and solving for a single variable in multiple variable equations, the researcher developed a test instrument. To develop the test instrument the researcher first examined the construct of study (i.e., solving for a single variable in

Table 7

Specific Student Matches

| Student Match (Abstract student versus CRA student) | School | Teacher | Age | | Grade | FCAT math Scores | | Semester Grades Abstract/ CRA | | Label | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Abstract | CRA | | Abstract | CRA | Abstract | CRA | Abstract | CRA |
| 1 | 1 | 1 | 12 | 12 | 7 | 3 | 3 | B | B | sld | sld |
| 2 | 1 | 1 | 13 | 12 | 7 | 4 | 3 | B | C | sld | sld |
| 3 | 1 | 1 | 12 | 13 | 7 | 7 | 6 | C | D | sld | sld |
| 4 | 1 | 1 | 12 | 13 | 7 | 3 | 3 | D | D | sld | sld |
| 5 | 1 | 1 | 12 | 13 | 7 | 2 | 2 | D | D | sld | sld |
| 6 | 1 | 1 | 12 | 12 | 7 | 6 | 7 | C | D | sld | sld |
| 7 | 1 | 1 | 13 | 12 | 7 | 4 | 5 | C | C | sld | sld |
| 8* | 1 | 1 | 13 | 12 | 7 | 7 | 6 | B | A | sld | sld |
| 9 | 1 | 1 | 12 | 12 | 7 | 3 | 3 | F | F | sld | sld |
| 10 | 1 | 1 | 13 | 13 | 7 | 4 | 4 | C | C | at-risk | at-risk |
| 11 | 1 | 1 | 13 | 13 | 7 | 4 | 3 | C | C | at-risk | at-risk |
| 12 | 1 | 1 | 12 | 12 | 7 | 3 | 3 | F | F | at-risk | at-risk |
| 13 | 1 | 1 | 12 | 12 | 7 | 4 | 4 | C | C | at-risk | at-risk |
| 14* | 1 | 2 | 13 | 12 | 6 | 7 | 6 | D | B | sld | sld |
| 15 | 1 | 2 | 11 | 12 | 6 | 6 | 6 | D | F | sld | sld |
| 16* | 1 | 2 | 12 | 11 | 6 | 6 | 5 | B | D | sld | at-risk |
| 17 | 1 | 2 | 12 | 12 | 6 | 4 | 4 | B | B | at-risk | at-risk |
| 18 | 1 | 2 | 11 | 12 | 6 | 4 | 5 | C | B | at-risk | at-risk |
| 19 | 1 | 2 | 12 | 11 | 6 | 2 | 3 | C | C | at-risk | at-risk |
| 20 | 2 | 3 | 12 | 12 | 6 | 5 | 5 | B | A | at-risk | at-risk |
| 21 | 2 | 3 | 12 | 11 | 6 | 4 | 4 | A | A | at-risk | sld |
| 22 | 2 | 4 | 12 | 12 | 6 | 5 | 5 | D | D | at-risk | at-risk |
| 23* | 2 | 4 | 11 | 12 | 6 | 4 | 4 | B | D | at-risk | at-risk |
| 24 | 3 | 5 | 12 | 12 | 6 | 3 | 4 | D | C | at-risk | at-risk |
| 25 | 3 | 5 | 12 | 13 | 6 | 4 | 4 | C | D | sld | at-risk |
| 26 | 3 | 5 | 12 | 11 | 6 | 5 | 5 | D | C | at-risk | at-risk |
| 27 | 4 | 6 | 13 | 13 | 7 | 2 | 2 | F | F | sld | sld |
| 28 | 4 | 6 | 13 | 12 | 7 | 4 | 4 | C | C | sld | sld |
| 29 | 4 | 6 | 14 | 13 | 7 | 6 | 5 | A | B | sld | sld |
| 30 | 4 | 6 | 13 | 13 | 7 | 5 | 4 | C | C | sld | sld |
| 31 | 4 | 6 | 13 | 14 | 7 | 3 | 3 | C | D | sld | sld |
| 32 | 4 | 6 | 13 | 13 | 7 | 2 | 2 | C | C | sld | sld |
| 33 | 4 | 6 | 12 | 13 | 7 | 3 | 3 | F | D | sld | sld |
| 34 | 4 | 6 | 13 | 13 | 7 | 3 | 3 | D | D | sld | sld |

* Despite semester grade or FCAT difference teacher reported equal overall performance.

algebra equations requiring transformations). To research each construct, the researcher reviewed the curriculum material supplied by each school for instructing students in that construct. From the curriculum material, the researcher developed a pool of 70 questions designed to assess the construct. The pool of questions was then distributed to four algebra teachers for expert review. The researcher asked the reviewers to provide feedback on each question. Reviewers then accepted the question, rejected the question, or provided improvements for each question. After receiving feedback from the reviewers, the researcher then made adjustments to the pool of items by eliminating or altering questions according to teacher feedback. With the revised pool of items, the remaining questions were then redistributed to the expert review panel for additional comments. Upon receipt of the updated changes, the researcher kept 63 questions and typed them on three sheets of paper. The 63-item pool of questions was then distributed to 32 students who had successfully completed his or her first year of prealgebra. Because most sixth-grade students have not been introduced multiple-step algebra equations, the 32 students used in developing the test were not matched to the students in the treatment and comparison groups. These 32 students were asked to complete all 63 questions to the best of their ability in an untimed testing situation. After completion of the pool of items, the researcher then analyzed the responses of the students according to whether the answer was correct or incorrect. The percent correct from all the students on each individual item marked the difficulty of the item. The 27 items that had a medium difficulty level between 0.375 and 0.625 correct were selected for the assessment instrument (see Table 8 for difficulty scores). Because of the length of time between tests and the expected pretest scores, the single test form was used for pretest, posttest, and

follow-up scores. Pretest measures were obtained 1 week prior to implementation of the treatment and comparison. Posttest measures were obtained upon completion of the last day of intervention, and follow-up measures were obtained 4 weeks after formal implementation had ended.

Table 8

Difficulty Score Analysis of Means for Building Assessment

| Question | N | Mean | Decision |
|---|---|---|---|
| 1 | 32 | 0.4688 | Keep |
| 2 | 32 | 0.6875 | |
| 3 | 32 | 0.6875 | |
| 4 | 32 | 0.6563 | |
| 5 | 32 | 0.3750 | Keep |
| 6 | 32 | 0.5312 | Keep |
| 7 | 32 | 0.5000 | Keep |
| 8 | 32 | 0.6563 | |
| 9 | 32 | 0.5625 | Keep |
| 10 | 32 | 0.5312 | Keep |
| 11 | 32 | 0.7187 | |
| 12 | 32 | 0.6875 | |
| 13 | 32 | 0.5937 | Keep |
| 14 | 32 | 0.5312 | Keep |
| 15 | 32 | 0.5625 | Keep |
| 16 | 32 | 0.5625 | Keep |
| 17 | 32 | 0.6250 | Keep |
| 18 | 32 | 0.6875 | |
| 19 | 32 | 0.5937 | Keep |
| 20 | 31 | 0.6129 | Keep |
| 21 | 31 | 0.6774 | |
| 22 | 31 | 0.6774 | |
| 23 | 31 | 0.6452 | |
| 24 | 31 | 0.6774 | |
| 25 | 31 | 0.5806 | Keep |
| 26 | 31 | 0.7097 | |
| 27 | 31 | 0.6774 | |
| 28 | 31 | 0.5806 | Keep |
| 29 | 31 | 0.6774 | |
| 30 | 31 | 0.6129 | Keep |

| Question | N | Mean | Decision |
|---|---|---|---|
| 31 | 31 | 0.7097 | |
| 32 | 31 | 0.6774 | |
| 33 | 31 | 0.7097 | |
| 34 | 31 | 0.6129 | Keep |
| 35 | 31 | 0.5484 | Keep |
| 36 | 31 | 0.6452 | |
| 37 | 31 | 0.6452 | |
| 38 | 31 | 0.6774 | |
| 39 | 31 | 0.4516 | Keep |
| 40 | 15 | 0.3333 | |
| 41 | 15 | 0.5333 | Keep |
| 42 | 15 | 0.5333 | Keep |
| 43 | 32 | 0.1875 | |
| 44 | 32 | 0.4375 | Keep |
| 45 | 32 | 0.2812 | |
| 46 | 32 | 0.3750 | Keep |
| 47 | 32 | 0.3437 | |
| 48 | 32 | 0.3125 | |
| 49 | 31 | 0.3548 | |
| 50 | 32 | 0.3750 | Keep |
| 51 | 32 | 0.1562 | |
| 52 | 32 | 0.3125 | |
| 53 | 32 | 0.3437 | |
| 54 | 32 | 0.2812 | |
| 55 | 32 | 0.3125 | |
| 56 | 32 | 0.3750 | Keep |
| 57 | 32 | 0.4375 | Keep |
| 58 | 32 | 0.2187 | |
| 59 | 32 | 0.4375 | Keep |
| 60 | 32 | 0.2812 | |

Additionally, a daily probe was developed for each student to observe his or her fluency (speed and accuracy) of skill acquisition after each lesson. Since algebra is an

abstract skill, a single use of a fluency probe will not mark when learning has been accelerated. Fluency probes are much more useful to a teacher by determining growth of number of items correct from consecutive lessons. Teachers can record when a student accelerates in a skill by observing fluency scores when the total number of correct answers increases greatly at the end of a lesson as compared to the previous lesson.

In order to develop these instruments, the researcher analyzed the overall construct into multiple short-term objectives. Short-term objectives were developed from multiple curriculum materials used by teachers from each school who participated in the study. Five objectives, listed in order, were as follows: reducing expressions, solving inverse operations, solving operations with negative or divisor variables, solving for variables on the same side of the equal sign, and solving for variables on opposite sides of the equal sign. The last two objectives involved transformations of equations. For each objective, a pool of 40 items taken from the curricula was constructed to match the corresponding lessons. Questions used in each pool did not match any question from the pretest, posttest, or follow-up test. Thus, tests items did not exactly repeat lesson items.

## Pretest Measures

Pretest measures were on the number of correct answers in the untimed test in transforming single variable algebra equations. Assessment of overall growth of each student compared pretest to posttest and follow-up scores. Additionally, pretest scores were useful in matching students. See Appendix A for the algebra pretest assessment.

Reducing expressions. As a precursor to solving for unknowns and performing transformations on equations, reducing expressions allows students to see that variables and numbers cannot be added or subtracted together. Additionally, different variables

must be handled differently. For example, the express ion 5X + 3B is already reduced to its most basic components until the values of B and X are revealed. Reducing expressions also adds instruction in negative numbers and negative variables. Although students in this study should have been previously introduced to negative numbers, using negatives in these lessons allows the teacher to check student understanding and reteach if necessary.

Inverse operations. Often considered one of the cornerstones of beginning algebra, solving for inverse operations, or, as often called, single variable equations, involves solving for an unknown number or set of unknown numbers. In order to help establish the concept that a variable represents a number, all variables in this unit represented one specific number. The usefulness of solving for single unknowns comes when students are solving questions when the answer can be computed with the information present. Students learned how to solve such equations using subtraction, addition, multiplication, and division to isolate the variable.

Negative and divisor variables. Solving for variables when they are negative or situated in the denominator of a fraction is not greatly different in practice from inverse operations outside of one transformation of the variable to make a positive sign or to make the variable positioned in the numerator. For example, when solving for 6/X = 3, the X first should be multiplied to each side to produce 6X/X = 3X. After dividing X by X, the student is left with a simple inverse operation 6 =3X. Since the student has already been instructed in solving for inverse operations, the student then carries on with what has already been learned.

Multiple variables on the same side of the equal sign. One of the more complex concepts that develops from solving for a variable is using the same techniques with multiple variables. Transforming equations allows the student to experience the ambiguity and abstractness associated with algebra (that variables need to be combined before solving). The usefulness of such a task becomes more evident in courses of study such as chemistry, physics, and personal finance and may be used for students to solve logic problems when they are not given sufficient information to precisely answer the question. When multiple variables are on the same side of the equal sign, then variables must be combined through addition or subtraction. Once combined, then the student is left with an inverse operation. Combining one side of an equation is simply reducing an expression.

Multiple variables across the equal sign. Transformations become more complex when variables are on both sides of the equal sign. In this step, students must add or subtract variables on one side of the equation in order to place all variables on the same side of the equal sign. Once the variables are on one side, students perform the same steps as they did in the previous step.

## Posttest Measures

Posttest measures followed implementation of single variable equations, transformations of multiple variable equations, and solving single variable equations using transformational skills. Posttest measures were also untimed and measured a variety of algebraic skills associated with solving for single variables in numeric and multiple variable equations. The algebra posttest assessment is the same instrument as used for the pretest (see Appendix A).

Follow-up Measures

The same measure as the pre- and posttest was used to assess the retention of what was learned for overall knowledge of algebra. Because the assessment measures overall algebra, each teacher covered graphing concepts (sixth grade) or fractions (seventh grade) during for the next few weeks after the posttest measure. These concepts do not overlap directly with solving for single variables using transformations. The algebra follow-up assessment uses the same assessment items as the pretest but in a different order (see Appendix B).

Fidelity

To ensure that the sequence of instruction and instruction components were used consistently throughout the treatment and comparison groups, a fidelity checklist was used for each teacher four times during the CRA instruction. For example, the researcher arrived in Teacher X's classroom randomly once during each phase of the CRA instruction to ensure the teacher was implementing each components listed in the teaching script as well as correctly demonstrating the use of the concrete or representational model. Additionally, the researcher also observed teacher X on 2 random days while implementing abstract instruction. The teacher was observed on delivery of instructional components, advance organizer, description of activity, modeling, guided practice, and independent practice. The treatment fidelity checklist is shown in Appendix C.

## Research Design

A pre-post-follow-up design with random assignment of clusters was employed for this study. Students were clustered by classroom and divided into two groups, a treatment group and a comparison group. The objective for both groups was to improve prealgebra skills. The sixth-grade teachers taught the comparison groups according to direct instruction following modeling, guided-practice, and independent-practice strategies. The teachers taught the treatment group also using direct instruction techniques, but with the addition of the CRA sequence of instruction. Since one-to-one matching is used in this analysis, the instruction is a within subject factor.

Table 9

Research Design

| Group | Procedures | | | | |
|---|---|---|---|---|---|
| Treatment | R | $O_1$ | $X_1$ | $O_2$ | $O_3$ |
| Comparison | R | $O_1$ | $X_2$ | $O_2$ | $O_3$ |

R = random clustered assignment, $O_1$ = Pretest, $X_1$ = Treatment instruction, $X_2$ = Comparison instruction, $O_2$ = Posttest, $O_3$ = 4-week follow-up.

## Procedures

In this section instructional procedures are described that outline the study. Materials used at each stage of the intervention are described as well as the stages of the procedural development. Included in the stages are a description of the pilot study, instructor preparation, and comparative descriptions of the intervention versus the control groups teaching.

The procedure was divided into five phases. The first phase was to test the CRA sequence model in order to determine the effectiveness of the program versus the traditional algorithm approach to algebra in a pilot study. This pilot study would enable close examination of the model in action exposing possible flaws to the model. The second phase involved a training session for teachers involved in the project. The third phase involved pretesting the students involved with the program. The fourth phase was the 19 lesson implementation of the program by the trained teachers. The final phase of the project was to test students' algebra knowledge in both a posttreatment test and a follow-up retention test. A post-treatment occurred as soon as the treatment was over while retention was measured 3 weeks following the posttreatment assessment. To maximize performance of the treatment, direct instruction was used during the implementation and pilot study phases. However, to eliminate direct instruction as a possible independent variable from the measurement, both comparison and treatment used direct instruction.

## Large N Design Study

The results of the pilot case studies (see Appendix D) show the CRA model as a possible intervention to help students scaffold their simple equation-solving processes to more complex equations. Additionally, both students in the pilot verbally remarked how important the CRA sequence of instruction was to their understanding and preparing them for future math challenges. To determine if the power of the CRA sequence could be used in large group settings, this study employed the use of whole classes to determine if students learn and retain algebra knowledge using the CRA sequence of instruction more efficiently than traditional forms of abstract instruction. To determine the

effectiveness of the CRA algebra sequence, it is important to describe the teacher training that occurred, the pretest conditions set, how the model was implemented and the follow-up assessments.

Teacher Training

Nine certified teachers in four middle schools in an urban county were trained to participate in the study. Each teacher learned from a training manual for both the CRA sequence of instruction and the algebra direct instruction comparison group. The manual consisted of scripted lessons to increase procedural reliability and corresponding lesson sheets to guide the students through instruction. Teachers were initially trained in a 1-day session consisting of introduction, modeling the delivery at each step, practicing lessons, and discussing how each teacher would be assessment in treatment fidelity. Additional tutoring of the model was provided individually for each teacher until she or he felt able to satisfy the components listed in each lesson script and comfortable with the CRA model. Treatment integrity was assessed during implementation through a checklist of the components. Integrity was scored according to percentage of checklist marks compared to master list. A 90% score or above was considered acceptable. Below 90% resulted in eliminating the teacher's data from the study. Scripted lesson formats are shown in Appendix E, and the corresponding learning sheets are shown in Appendix F.

Pretest Conditions

Subjects were matched according to their previous course in math, chronological age, grade level, pretest score, previous math achievement score, semester math grade scores, and disability or at-risk label. Teachers who participated in the study taught two

sixth-grade math classes that the teacher indicated had similarly performing students to help with the matching criteria. Using a coin flip, the researcher then randomly chose one of the two math classes for the teacher to teach using the CRA sequence and the other class to be taught using the abstract-only traditional methods.

Treatment Group

Students in the treatment group worked in the same classroom setting that they had throughout the year, but now their teachers taught class using the CRA sequence of instruction. Since the students had minimal prior experiences with algebra, the CRA sequence of instruction would introduce them to the many concepts and abstract thinking associated with algebra. Each CRA lesson lasted through four steps. These four steps were used for instruction at the concrete, representational, and abstract stages of instruction.

Step 1: Introduce the lesson. To introduce the lesson, the instructor would state what students were going to do as well as explain to each student why they were going to learn the new skill. To ensure that motivation was equally delivered to students, the rationale for the lesson and advance organizer were part of each scripted lesson given to teachers.

Step 2: Model new procedure. To model the procedure of each concept, the instructor would think aloud the processes needed to solve or complete the problem. Problems used in these examples were not only included in the instructor's manual but also the students' handout. While the instructor voiced aloud a problem, students were required to follow along by writing the steps to the problem on their handout. When using manipulatives, the teacher showed the objects on the overhead. For each lesson,

the instructor would model one or two problems without student input. An example of the think aloud would be as follows:

ISOLATE--Isolate the variable; subtract, add, multiply, and then divide out other numbers; organize the equation to balance; let calculations fly by first; answering equation on the variable's side; and then totaling the equations on the other side and finishing the question by evaluating whether or not the answer makes sense.

Step 3: Guide students through instruction. After the procedures have been modeled, it was time for the students to attempt answering the questions. Students then were to work individually to solve for the problems under the guided instruction section of their handout. In this section, the instructor would have students answer each step to the problem individually while rotating about the room to check on student understanding. Upon observation that students were mastering the steps to solve the problems, the teacher would then indicate to the students that it was time to work independently.

Step 4: Set up independent practice. Once the students showed the ability to move through the steps under teacher guidance, they were then instructed to attempt answering question on their handout independently. This independent instruction included six to eight questions of varying difficulty that students must answer without teacher assistance. If students had a question, naturally they were to ask for help, but the teacher was not supposed to give them answers, only to question the student through the steps needed to solve the problem.

Comparison Group

Although teachers may teach their classes using a variety of techniques, the comparison instruction had to be controlled. Using the researched effective instructional techniques of introducing a lesson, modeling a procedure, working with students through guided practice, and then independent practice before assessment, the comparison instruction was set. This sequence of instructional steps matched the treatment instructional steps. This enabled the researcher to isolate the effectiveness of the CRA sequence and materials as the focus of the research question. Since the students had minimal prior experiences with algebra, the abstract approach of instruction would introduce them to the many concepts and abstract thinking associated with algebra. Students in the comparison group worked in the same classroom setting that they had throughout the year. Each lesson lasted through the same four steps as the comparison group. These four steps were used for each component of instruction.

Step 1: Introduce the lesson. To introduce the lesson, the instructor would state what students were going to do as well as explain to each student why he or she would be learning the new skill. To ensure that motivation was equally delivered to students, the rationale for the lesson and advanced organizer were part of each scripted lesson given to teachers.

Step 2: Model new procedure. To model the procedure of each concept, the instructor would speak aloud the thought processes needed to solve or complete the problem. Problems used in these examples were not only included in the instructor's manual but also the students' handout. While the instructor voiced aloud a problem, students were required to follow along by writing the steps to the problem on their

handout. For each lesson, the instructor would model one or two problems without student input. An example of the think aloud would be as follows:

ISOLATE--Isolate the variable; subtract, add, multiply, and then divide out other numbers; organize the equation to balance; let calculations fly by first; answering equation on the variable's side; and then totaling the equations on the other side and finishing the question by evaluating whether or not the answer makes sense.

Step 3: Guide students through instruction. After the procedures have been modeled, it was time for the students to attempt answering the questions. Students then were to work individually to solve for the problems under the guided instruction section of their handout. In this section, the instructor would have students answer each step to the problem, step by step ensuring their progress by rotating around the room. After observing that students had successfully completed the steps to solve the problems, the teacher would have students start their independent work.

Step 4: Set up independent practice. Once the students showed the ability to move through the steps under teacher guidance, they were then instructed to attempt answering question on their handout independently. This independent instruction included six to eight questions of varying difficulty that students must answer without teacher assistance. If students had a question, naturally they were to ask for help, but the teacher was not supposed to give them answers, only to question the student through the steps needed to solve the problem.

## Materials

The materials used in the study were developed to determine the difference in acquisition and retention of algebraic understanding for students with learning disabilities

in math. Students in both the treatment and the comparison groups used the assessment instrument pre-, post-, and to follow-up instruction in order to assess differences between the matched pairs. Treatment and comparison groups also used the same problems on similar daily learning sheets to guide them and their teachers through instruction. See Appendix F for the abstract comparison learning sheets and treatment CRA learning sheets. The comparison CRA group had an addition of materials to the instruction for both concrete and representational instruction.

Concrete. Students who participated in the treatment group used concrete objects to represent parts of expressions and equations. Yellow colored paper cut into 2-inch by 2-inch squares with black scripted letters represented variables or unknowns. Popsicle sticks and flat toothpicks represented tens place value and ones. One-ounce plastic cups represented coefficients, and a 10-inch string represented the equal sign. Reversible paper with green on one side and red on the other represented addition and subtraction signs. Each addition/subtraction display was on a 1.5-inch by 1.5-inch square cardboard cutout. On the green side of the paper the addition sign was displayed, and on the red side the subtraction sign was displayed. Division was displayed as a fraction with a 6-inch blue strip of paper representing the fractional divisor.

Representational. Students in the treatment group also used pictorial representations of the concrete step to represent parts of the two-variable equations. Worksheets with circles and lines on them showed students the connection to the concrete stage. The circles corresponded to the number and variable sets, while vertical lines marked the tens place, and shorter diagonal lines represented one place numerals. Variables were simply marked by their corresponding letter (e.g., X, Y, K). Staggering

vertical lines represented equal signs. As students progressed through the representational stage, circles were excluded, and specific variables began to be more random, as part of scaffolding student understanding to prepare for different equations. Word problems used followed the same steps using representations as the numeral problems.

Abstract. Students in both the treatment group and control group solved two-variable algebra equations at the abstract level. Worksheets with numerals only and space to solve the problem indicated the abstract stage. The space provided allowed students to draw representations as needed, but the goal of this stage was to move beyond written representations and into written numerals to solve problems.

The concrete and representational steps were set differently than abstract instruction to determine if breaking down the expressions and equations by students aided their learning. It was essential that the representations matched the concrete objects and that the representations had a logical flow into abstract numerals. To test the effectiveness of the flow designed in this CRA model, a specific procedure for measuring outcomes was designed.

## Experimental Design and Analysis

Teacher training occurred before implementation. Following training, teachers were informally assessed on their knowledge of the instructional programs they were delivering, including the CRA sequence, the A algorithm instruction, and direct instruction in general. Treatment integrity also was assessed through a checklist recorded by the primary researcher during teacher implementation.

The experimental design employed in this study was repeated measures with two within factors: type of instruction and test occasion. Subjects were matched and assigned to level of instruction--either the control group or the treatment group. Subjects were tested before the implementation phase. This score provided a starting point for judging the repeated measures growth upon the posttreatment and the retention scores. The posttreatment score was assessed the next day following implementation of the final lesson. Additionally, students were assessed 3 weeks following posttreatment to analyze retention.

The dependent measure, number of correct answers on algebra assessment out of 27 possible, was analyzed for both instruction groups for pretreatment, posttreatment, and retention. The repeated measures analysis of variance was used to determine if any significant differences existed between the instructional groups on posttreatment and retention measures. Because students were matched on pretest assessment, the pretreatment scores did not significantly differ. Follow-up univariate analyses of variance and t-tests were computed for acquisition and retention. A 95% confidence level was established for rejection of the null hypothesis.

# CHAPTER 4
# RESULTS

## Introduction

The purpose of this study was to compare the effects of traditional abstract instruction with a newly developed CRA sequence of instruction on initial instruction of algebra equations. Students with learning disabilities in math and those at risk for future failure in mathematics at the sixth- and seventh-grade levels participated in this research. The primary question was as follows: What are the effects of the CRA sequence of instruction versus traditional abstract instruction on the acquisition and retention of algebra equations?

To answer this research question, the posttest and a 3-week follow-up performance of the two groups were compared. The treatment group received CRA instruction on four concepts in initial algebra over a 19-lesson sequence culminating in instruction on solving for single variables using multiple step transformation equations. The comparison group received traditional abstract instruction over the same concepts and corresponding 19 lessons, also culminating in instruction on solving for single variables using multiple step transformation equations. Since teachers taught each group at least once throughout every school day, students were matched across classrooms.

After matching students across teacher's classes according to disability at-risk label, age within 1 year, grade level, similar semester performance, FCAT math score within one stanine, and pretest score, the independent variable was reduced to type of

instruction--treatment or comparison. This chapter provides the results of statistical analyses of the data from this study and displays individual student results (see Table 10). Additionally, treatment integrity data are revealed.

## Overall Findings

In this section the researcher determined any statistical differences between the CRA sequence of instruction and traditional abstract instruction for initial algebra equation instruction for sixth- and seventh-grade students after reducing the chance of error due to previous learning, age, grade, teacher, and future mathematical concerns. Repeated measures analysis of variance was performed on two levels of instruction (CRA versus abstract) and three levels of occasions (pretest, posttest, and follow-up). See Table 11 for analysis of variance summary. The students who participated in CRA instruction outperformed their traditional abstract instruction peers on the posttest and follow-up test. See Table 12 for descriptive statistics. Both groups, however, showed significant improvements in answering single variable algebraic equations from the pretest to the posttest and the pretest to the follow-up. Additionally, treatment integrity measures showed that teachers consistently were able to perform the CRA instruction as specified in scripts.

## Findings by Repeated Measures

Repeated measures analysis revealed that there was a significant main efect difference between test scores of students who participated in the CRA sequence of instruction and test scores of students who participated in traditional abstract instruction, $F(1, 33) = 29.982$, $p = 0.000$). There was a significant difference between both groups'

Table 10

Individual Matching Data

| Student Match (Abstract versus CRA) | Pretest Scores | | Posttest Scores | | Follow-up Scores | |
|---|---|---|---|---|---|---|
| | Abstract | CRA | Abstract | CRA | Abstract | CRA |
| 1 | 0 | 0 | 2 | 0 | 8 | 0 |
| 2 | 0 | 2 | 0 | 3 | 7 | 3 |
| 3 | 0 | 0 | 0 | 4 | 0 | 18 |
| 4 | 0 | 0 | 5 | 16 | 4 | 14 |
| 5 | 0 | 0 | 2 | 4 | 8 | 5 |
| 6 | 0 | 0 | 9 | 5 | 16 | 17 |
| 7 | 0 | 0 | 7 | 5 | 9 | 5 |
| 8 | 0 | 2 | 17 | 21 | 21 | 22 |
| 9 | 0 | 0 | 0 | 5 | 0 | 9 |
| 10 | 0 | 0 | 0 | 6 | 4 | 6 |
| 11 | 0 | 0 | 1 | 9 | 1 | 4 |
| 12 | 0 | 0 | 0 | 5 | 2 | 9 |
| 13 | 0 | 0 | 1 | 12 | 0 | 10 |
| 14 | 0 | 0 | 9 | 11 | 3 | 16 |
| 15 | 2 | 0 | 9 | 11 | 4 | 7 |
| 16 | 0 | 0 | 14 | 23 | 15 | 21 |
| 17 | 0 | 0 | 0 | 14 | 1 | 4 |
| 18 | 0 | 0 | 4 | 9 | 2 | 11 |
| 19 | 0 | 0 | 0 | 4 | 0 | 2 |
| 20 | 0 | 0 | 0 | 3 | 2 | 1 |
| 21 | 0 | 0 | 0 | 4 | 2 | 1 |
| 22 | 0 | 1 | 1 | 6 | 1 | 3 |
| 23 | 0 | 0 | 6 | 13 | 9 | 10 |
| 24 | 0 | 0 | 0 | 0 | 0 | 3 |
| 25 | 0 | 0 | 2 | 7 | 0 | 2 |
| 26 | 0 | 0 | 2 | 9 | 0 | 9 |
| 27 | 0 | 0 | 0 | 2 | 0 | 0 |
| 28 | 0 | 0 | 5 | 6 | 3 | 5 |
| 29 | 0 | 0 | 7 | 11 | 4 | 4 |
| 30 | 0 | 0 | 1 | 7 | 0 | 0 |
| 31 | 0 | 0 | 0 | 3 | 0 | 0 |
| 32 | 0 | 0 | 0 | 8 | 0 | 6 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 |
| 34 | 0 | 0 | 0 | 3 | 0 | 0 |

Note. These scores are based on a maximum test score of 27.

Table 11

Summary of Repeated Measures ANOVA for instruction and Test Occasion

| Source | Type III SS | df | MS | F |
|---|---|---|---|---|
| Instruction | 301.490 | 1 | 301.490 | 29.982* |
| Error (Instr) | 331.843 | 33 | 10.056 | |
| Test Occasion | 1180.480 | 2 | 590.240 | 31.979* |
| Error (Test) | 1218.186 | 66 | 18.457 | |
| Instr x Test | 157.775 | 2 | 78.887 | 13.888* |
| Error (Instr x Test) | 374.892 | 66 | 5.680 | |

* Significant at the p<0.05 level.

Table 12

Descriptive Statistics Within Each Instructional Group and Test

| Descriptive Statistics | | | | | |
|---|---|---|---|---|---|
| | N | Minimum | Maximum | M (out of 27 possible) | SD |
| PRETEST CRA | 34 | .00 | 2.00 | 0.1176 | 0.4093 |
| PRETEST Abstract | 34 | .00 | 2.00 | 0.0588 | 0.3430 |
| POST CRA | 34 | .00 | 23.00 | 7.3235 | 5.4812 |
| POST Abstract | 34 | .00 | 17.00 | 3.0588 | 4.3689 |
| FOLLOW CRA | 34 | .00 | 22.00 | 6.6765 | 6.3232 |
| FOLLOW Abstract | 34 | .00 | 21.00 | 3.7059 | 5.2080 |
| FOLLOW Abstract | 34 | .00 | 21.00 | 3.7059 | 5.2080 |
| Valid N (listwise) | 34 | | | | |
| Valid N (listwise) | 34 | | | | |

Note. Eta = 0.237; Eta squared = 0.056.

pretest, posttest, and follow-up test scores, $F(2, 66) = 31.979$, $p = 0.000$). There was also an interaction effect between the type of instruction and the test occasion, $F(2, 66) = 13.888$, $p = 0.000$). This interaction effect indicated that different instructional groups performed significantly different across the pretest, posttest and follow-up test.

Instructional Differences

The interaction between test occasion and treatment condition yielded a significant difference, $F(2, 66) = 13.888$, $p = 0.000$. This significant score ($p<0.05$) shows that students who received CRA instruction scored differently upon test occasion than their traditionally taught matches. Follow-up tests revealed specifically where the differences exist. Since three follow-up tests were used to determine areas of significant differences between groups, the Bonferroni correction procedure was used to maintain a 95% confidence level. Thus, the alpha error level was divided by the number of follow-up tests, three ($\alpha = 0.05 / 3 = 0.0167$). For the comparisons to show a significant difference, the alpha error level needed to be below 0.0167.

Running additional $t$-tests on group means indicated the students who received CRA instruction over the 4-week intervention outperformed matched students who received traditional instruction during the same time period with the same teacher. While there was no significant difference between pretest scores, $t(66) = 0.63$, $p = 0.268$) between the two groups, there were significant differences between the posttest and follow-up scores. The group who learned through CRA instruction scored a mean of 7.324 ($SD$=5.481) out of 27 possible, outperforming the group who learned through abstract instruction with a mean of 3.029 ($SD$=4.387) out of 27 possible. The difference

between the means of 4.029 accounted for a significant difference, $t(66) = 6.46$, $p = 0.000$). The group that learned through the CRA sequence of instruction also outperformed the abstract group on the 3-week follow-up test, $t(66) = 3.28$, $p = 0.001$). The CRA group scored a mean of 6.68 ($SD = 6.32$) out of 27 possible on the follow-up test, while the abstract group scored a mean of 3.71 ($SD = 5.21$) out of 27 possible.

While significance between groups was of importance to this research, the high standard deviations within each group's specific test scores was also of importance. Standard deviations were fairly high relative to mean scores for the pretest and posttest. Interestingly, SDs increased from the posttest to the follow-up test despite no significant growth difference between each group's posttest and follow-up test. These high variances of scores indicated that, although students selected for the research had learning disability labels and at-risk concerns, they were highly variable in the amount of learning during this program.

Test Difference by Group

The initial instruction in the treatment and comparison groups combined showed significant improvement in students' ability to solve single variable multiple step algebra equations, $F(1, 33) = 31.98$, $p = 0.000$). Post hoc analysis of variance of each instructional group showed that the comparison abstract group improved from pretest to posttest and pretest to follow-up, $F(2, 99) = 8.34$, $p = 0.000$, and the treatment CRA group improved performance from pretest to posttest and pretest to follow-up, $F(2, 99) = 23.10$, $p = 0.000$. The mean scores of the comparison group increased from 0.059 on the pretest to 3.059 on the posttest and then 3.706 on the follow-up exam out of 27 possible correct items. The mean scores of the treatment CRA group increased from 0.118 on the

pretest to 7.324 on the posttest and 6.680 on the follow-up test. While both groups showed a significant increase from pretest to posttest and pretest to follow-up exam, neither group showed any significant difference between their respective posttest and follow-up exams. These significant differences illustrated the positive effects of CRA instruction for initial learning of algebra.

Error Pattern Analysis for Each Group

Examining the answers on tests and daily lessons not only allows for inspection of more than right or wrong answers but also the details of why a student might have solved for a variable incorrectly. While errors were similar between groups, the difficulty with negative numbers was more evident from the abstract group. Additionally, combining like variables was more problematic for the abstract group. Both groups showed difficulty solving for variables on the same side of the equation and not completing calculations before solving for the variable.

Solving for negative numbers, whether in a multiplication or division problem or an addition or subtraction problem, can lead to an incorrect answer. Many teachers complain about the difficulty of students understanding the concept of negative numbers. Likewise, teachers have difficulty explaining and showing negative numbers and related use of negative numbers. It is easier to explain subtraction of two negative numbers than to explain multiplication of two negative numbers. For example, a person losing $3, then spending $2 more for a total loss of $5 is equivalent to –5. On the other hand, negative 3 groups of losing $4 is equivalent to –12. There are few effective ways to explain negative groups.

Students had difficulty solving for variables when terms needed to be subtracted or added out to combine with the other side of the equal sign. For example, in an equation such as $5x + 6 = 2x$, the 2x and the 5x need to be combined. One way would be to move the 5x to the right side of the equal sign by subtracting it from the left side, such as $5x - 5x + 6 = 2x - 5x$. This is the more efficient way to combine the like variables. Students often are unable to solve for the variable on the right side because of confusion. Similar to difficulties with multiplication tables where $5 \times 3 = 3 \times 5$, some students have difficulty seeing that $6 = -3x$ is the same equation as $-3x = 6$. Therefore, instead of the most efficient means of combining the variable, the student subtracts the 2x. In this case, the student transforms the equation to $5x - 2x + 6 = 2x - 2x$. If the student knows not to combine the 6 with the variables on the left side, then the student is left with $3x + 6 = 0$. While some students solved here, others knew to subtract the 6 first. In this case, subtraction is used not once, as in the previous maneuver, but twice. Increasing the number of operations made to solve an equation may increase the likelihood of producing an error.

Both groups had difficulty solving for the variable when numbers were on the same side of the equal sign as the variable. For example, the student would answer the equation $3x - 6 = 0$ as $x = -2$ or when confused with the negative number concept 2. While these confusions often may lead to the correct answer when combined errors are made, the student's answers are inconsistent with processes and steps to solving equations.

Solving for variables correctly or incorrectly may be the ultimate purpose, but error pattern analysis should not be ignored. When errors such as these are found

repeatedly, then future work with solving equations should take into account the potential trouble spots. In this research the trouble spots usually resulted in incorrect answers, but a few times students became lucky with a correct answer due to multiple errors that coincidentally led to the actual solution.

## Treatment Integrity

Before instruction began, one sixth-grade teacher dropped out of the project due to personal concerns over the concepts. Treatment integrity was taken across six teachers (see Table 13). After observation of the lesson of a seventh-grade teacher, the teacher opted out of the project stating that her students did not know their multiplication tables and could not add or subtract negative numbers without difficulty. The other six teachers were observed four times while delivering treatment instruction. While the observations did not address teacher enthusiasm, behavior management, or other possible performance-altering teacher actions, teachers were recorded as to the components of the lessons delivered. All participating teachers received 100% in delivery of an advance organizer, introduction, modeling, guided practice, and independent practice. Since most teachers complained that the use of probes was too time consuming along with each lesson, all but three teachers eliminated the use of the daily probe after the third lesson. The other teachers used probes for marking student progress proceeding abstract lessons in the CRA sequence and corresponding lessons in the comparison group. In addition to observations, the researcher conducted weekly discussions regarding both treatment and comparison groups instruction to answer questions the teacher would have about the next week's lessons.

## Summary

Both groups of students showed significant learning from the pretest to the posttest scores, however, score increases from pretest to posttest and follow-up were not associated across treatment and comparison. Scores were highly variable among both groups in posttests and follow-up tests. Not all students benefited the same degree from either CRA treatment or abstract comparison. Although both groups were not significantly different in pretest scores, the students who participated in the CRA sequence of instruction outperformed the students who participated in traditional abstract instruction on the posttest and follow-up exams.

Table 13

Treatment Integrity Scores for Each Teacher

| Teacher | Advance Organizer | Intro | Model | Guided Practice | Independent Practice | Total |
|---|---|---|---|---|---|---|
| 1 | 4/4 | 4/4 | 4/4 | 4/4 | 4/4 | 20/20 |
| 2 | 4/4 | 4/4 | 4/4 | 4/4 | 4/4 | 20/20 |
| 3 | 4/4 | 4/4 | 4/4 | 4/4 | 4/4 | 20/20 |
| 4 | 4/4 | 4/4 | 4/4 | 4/4 | 4/4 | 20/20 |
| 5 | 4/4 | 4/4 | 4/4 | 4/4 | 4/4 | 20/20 |
| 6 | 4/4 | 4/4 | 4/4 | 4/4 | 4/4 | 20/20 |

CHAPTER 5
DISCUSSION

## Introduction

The theoretical and future research implications as well as limitations of this research are explained in this chapter. While this research contains statistically significant results, other aspects of this project are significant. In this section these strengths and limitations, according to teacher and student comments, are discussed.

## Summary of Hypotheses and Results

Before the program was implemented, two null hypotheses were set to mark research expectations. These hypotheses were then tested using the posttest and follow-up scores. The hypotheses were as follows:

$Ho_a$: There will be no significant difference on measures of solving for a single variable in an algebra equation requiring transformations between students receiving CRA instruction and those who receive abstract instruction following instruction.

$Ho_b$: There will be no significant difference on measures of solving for a single variable in an algebra equation requiring transformations between students receiving CRA instruction and those who receive abstract instruction obtained 3 weeks after instruction has ceased.

Both null hypotheses were rejected. There was a significant difference between the instructional groups on the posttest and on the follow-up test. Overall differences,

differences by occasion, and differences by occasion per group were revealed by using a repeated measures analysis of variance with post hoc t-tests to specifically determine the points where differences occurred. The students who received CRA instruction outperformed their traditionally taught peers on the posttest and follow-up test.

## Theoretical Implications of the Research Findings

As mentioned in Chapter 2, constructivistic math is a controversial topic that some researchers support and others reject. The concern with constructivistic math is that students may not receive proper teacher modeling and guidance. Clements and McMillan (1996) claimed that often student-constructed programs harm student growth when the students are not informed of the processes behind their work with manipulatives. On the other hand, students need to come to constructions of knowledge when abstract concepts are introduced. Teachers refer to such spontaneous constructions of knowledge (Piaget, 1963) as "aha's" when a student discovers a deeper concept within a set of mathematical problems. Because of the concerns with implicit instruction though, the instruction needs to be designed so that student's conceptual knowledge constructions are manipulated by the teacher.

In this study, explicit instruction was used in both forms of instruction. Because the concepts of algebra involve abstractness, students are left to develop their interpretation of conceptual knowledge. By implementing the CRA sequence of instruction, abstractness is simplified through enactive and iconic forms of representation (Bruner, 1983). Thus, while both forms of instruction use constructivistic principles, there were essentially two levels of explicitness within exogenous constructivism (Moshman, 1982). The CRA sequence of instruction involved more explicit instruction

than the traditional form of instruction that also revolved around direct instruction principles.

The significant difference shown between using manipulatives and pictorial representations versus traditional abstract instruction with the same teacher modeling and guided practice leads us to believe that manipulatives may allow students to further examine the teacher's instruction and create their own proper personal interpretations of how to solve algebra equations. The apparent success of this procedure implementing exogenous constructivism stresses more explicit instruction before students begin to make their own constructions and connections within algebra. Thus, this research supports explicit and direct instruction principles within exogenous constructivism.

## Practical Implications of the Research Findings

Teachers need to use concrete and pictorial representations that are appropriate to the age level and content level of the students. Howard, Perry, and Conroy (1995) noted that many teachers in secondary settings do not use manipulatives with their students. Some teachers may not trust the usefulness or efficiency of manipulatives for higher level algebra. For this reason, the most popular algebra model targets elementary schools for its product. This may be because the program operates effectively only with inverse operations because of its flaw in representing the coefficient. Additionally, some teachers may not wish to use manipulatives at higher levels because of the fear that students may mindlessly solve problems and not truly understand the concept. This program, however, does represent the coefficient in a manner that can translate to higher conceptual learning. Additionally, this algebra model does not appear to oversimplify equations, forcing students to understand concepts before solving for variables. Teachers

need to continue using manipulatives and pictorial representations after elementary school, but researchers and publishers also need to produce materials that enable concepts to be disseminated correctly at secondary levels. This program shows promise to allow teachers to teach higher level concepts in a manner that students can be active in solving and even take home with them to use outside of the classroom.

The power of the CRA sequence of instruction is also supported by this research. The CRA sequence of instruction has been beneficial to students with disabilities and academic difficulty to learn basic facts (Harris, Miller, & Mercer, 1995; Mercer & Miller, 1992) and initial fractions (Jordan, Miller, & Mercer, 1999). Research even supports the CRA sequence to represent word problems in simple algebraic inverse operations (Maccini & Hughes, 2000). Prior to this research, there has not been a published examination of a manipulative and pictorial method that translates into more complex equations beyond simply solving for single inverse operations. Not only did this research show how the CRA sequence of instruction could be applied properly to coefficients other than one, but also this research showed how proper materials and pictorial representations could be used to perform algebraic transformations.

Some programs for students have been effective in individual instruction (Maccini & Hughes, 2000) but need explanation as to how teachers can use the program for an entire class. The usefulness of this CRA sequence is not only that it can be effectively delivered in a class situation but also that it can be used in an inclusion class situation as well. While students with disabilities have not always been successful academically in mainstreamed settings (Zigmond & Baker, 1995), this program appeared to help some students with math disabilities. While this program does show effectiveness for students

with disabilities and math difficulties to be taught with normally achieving peers, the results reported in this research do not assess the power of the program on high- or normal-achieving students.

## Limitations to the Present Study

There were a few limitations to this study. The first limitation revolved around the assessment. Other minor limitations involved the fixed-step approach to the sequence of lessons, motivational differences between teachers, and the delay in completion of the project beyond the set 4 weeks. While these limitations may have affected the assessment scores of the project, none of them should have significantly affected the differences between treatment and comparison scores.

The assessment used for pretest, posttest and follow-up test in this project was made specifically for this project. To measure the differences between traditional instruction and CRA instruction, the assessment was designed to be difficult enough to reduce the chance of ceiling effects while measuring the end result of students advancing through all 19 lessons. To develop the appropriate questions, the assessment only included items that were of medium difficulty to students who had passed Algebra 1 and thus had been instructed in the concepts taught during this project. The questions on the posttest and follow-up tests did not cover the spectrum of the five lessons taught but rather the final, most difficult step. Since most students do not learn the algebraic concepts taught during this project until they are in ninth grade, the sample of students used to develop the assessment differed from the targeted sample of students with learning disabilities in the study. Additionally, the assessment was not standardized to the entire county but merely reflected the knowledge of 32 students mixed between

public and private school who had completed Algebra 1 with an A or B grade. Only eight of the students had been diagnosed with learning disabilities. The same assessment items were used in the pretest, posttest, and follow-up tests. While there may be a concern with history effects, most teachers claimed that it would be uncharacteristic of their students to remember the specific items from one test to the other. While assessment items were a concern, grading the items became an additional concern.

One teacher commented about the grading of the assessments. He commented that there were differences between the two groups beyond simple right and wrong answers as scored in this project. He explained that students who received CRA instruction appeared to attempt more problems, were more enthusiastic about learning algebra, and followed more steps in their procedures to answer for variables despite final right or wrong answers. While the research used correct versus incorrect answers on the assessment rather than a rubric marking a student's attempt at proper steps, error pattern analysis was used in the results section. Error pattern analysis along with final solutions brought a more full picture of students' performance solving multiple step algebra equations.

The delay due to the anniversary of Columbine and many end-of-the-year activities led to a late end of the project. As a result, the last lesson and posttests were given approximately 1 week later than planned; thus, the follow-up test was given to students 1 to 2 weeks before the end of the school year. Teachers commented that students who were typically low achieving (i.e., those involved in the project) exhibited apathy near the end of the year, and without extra motivation to try on the follow-up test, some students gave minimal effort.

This project employed the sequence of introduction of lessons according to algebra textbook publishers (i.e., Prentice Hall, McGraw-Hill). Many other algebra textbooks exist. Many of the low performances of students overall for this grade level may be due to the sequence of lessons. Matching reduced the possibility that the curriculum lesson sequence influenced the comparison of the treatment and comparison group. However, the curriculum does influence how well students perform overall. Additionally, this project lasted longer than the typical 2-week math unit. This made motivation efforts difficult for teachers who enjoy change ups and altering the pace and sequence of lessons based on teacher observation.

One teacher at a school where multiple teachers participated in the project stated that different teachers set different motivations to the program. Since the Institutional Review Board (IRB) did not allow the project to affect grades, some teachers had to set motivation through teacher satisfaction while others attached extra credit for effort and success. Two teachers worked with students to do their best merely to please the teacher. Two teachers used extra credit as the incentive to perform. Two other teachers attached no obvious class-wide extrinsic motivation asking students to try their best but that results would not affect their grades in any way. As a result students worked to meet different motivators depending on the teacher's installment of motivation and the students' relationship with the teacher.

The assessment, sequence of lessons, and delay in project completion may have led to a reduction of perceived student performance on the posttest and follow-up test. However, these limitations affected each class similarly. Since the data analyses were compared across matched subjects, the difference between how these limitations affected

each member of a matched pair should be negligible. However, some the suggestions of teachers and students as well as observations throughout the project help prepare future research on the CRA sequence of instruction and research on algebra instruction.

## Implications for Future Research

Throughout this project, teachers commented on the effectiveness of the model. However, teachers' positive comments and claims of pleasant surprises about students' performance decreased when equations became more difficult (i.e., variables were presented on both sides of the equal sign). Additionally, five of the six teachers commented on the students' inability to solve equations when adding and subtracting negative numbers were involved. Talking with students at the end of the project reiterated these concerns.

Specifically to the CRA model, two teachers commented that the hands-on program resulted in student success that was not accounted for in the assessment provided. One teacher remarked that the assessment covered only the most difficult material and not the other four types of equations and expressions learned. While this limitation had to do with the assessment development, these positive comments regarding the CRA model showed teacher support for teaching algebra to students at the sixth- and seventh-grade levels. While there were clear differences in performance between students who participated in CRA instruction with those who participated in traditional instruction, an encompassing study of the variables with algebra for these age levels needs to be examined.

There was no notable difference between the overall mean scores of the six-grade students versus the seventh-grade students. This introduces the possibility that algebra

can be introduced at the sixth-grade level. This finding supports Demby's (1997) work with six graders in Poland on algebra instruction. If students may begin to understand algebra work at the sixth-grade level, then textbooks may include introductory aspects of algebra early on in middle school. Before the inclusion in six-grade textbooks, though, more research needs to focus on prerequisite skills to algebra to determine if students are prepared for initial instruction.

Introduction to algebraic conceptual knowledge may be too difficult for some students at these grade levels. While some students made great strides between pretest scores and posttest and follow-up test scores, others made no gain. This lack of growth after 4 weeks of instruction is disparaging although many teachers appeared unmoved by the findings. More work is needed to determine what skills in arithmetic are needed prior to algebra instruction to ensure student capability. Until we know, more programs, such as the present one, should be activated. Such programs provide initial instruction in algebra while supporting past learning in basic arithmetic.

One teacher explained that he felt it was unfair for only one group of his students to receive the treatment since the power of the treatment was so effective. This moral dilemma has to do with the research design. A possible solution to this design would be to reverse the academic instruction and reteach the comparison group using CRA. There needs to be further investigation as to whether this CRA model should incorporate concrete manipulatives throughout every concept in algebra before working with representations. Representation to abstract instruction needs to be examined. Additionally, because this was a whole-class strategy, there was no investigation of when students reached mastery at any level. In such cases, mastery may not have been reached

before students moved to the next lesson. In a nonexperimental situation, teachers would be better able to adjust lessons and amount of learning to their class. In this project, however, the lessons were fixed-step instruction, sometimes possibly pulling students through the lessons.

This study dealt with initial instruction of algebraic concepts. While there is evidence that this CRA model for algebra may be effective for students with learning disabilities in an urban county, there is only theoretical evidence that this model will be an effective intervention to correct errors in previous learning. Further investigation is needed to assess the possibility of this CRA model as a corrective intervention.

## Summary

This algebra research on the CRA sequence of instruction brings continued insight into the effectiveness of hands-on and pictorial representations for complex mathematics. The students in this project who were taught using CRA sequence of instruction performed better on posttests and follow-up tests while committing fewer errors with negative numbers and transforming equations before solving for the variable. This research contributes to the growing understanding of algebra instruction for students with learning disabilities or those at risk for failure in secondary mathematics. Continued research as to how to help students with disabilities understand algebra concepts will help graduation rates and should improve students' ability to think abstractly.

APPENDIX A
PRETEST AND POSTTEST ASSESSMENT

Name: ______________________________ Date: ______________________

Year in School: _______________________ Math grade on last report: _______

Present Math Class: ___________________ Teacher's last name: ___________

Previous Math Class: __________________

Solve for the variable. Circle your answer and write legibly.

ex. 5N = 15
N = 15/5
N = 3

6 – 2X = X - 6

7 – 3X = 4X + 14

6Y – 3 = 3Y - 9

5X – 8 = 2X - 7

-8X + 7= 22 - 3X

5B + 8 = 3B + 26

5Y + 9 = 8Y – 15

-4N + 12 = 21 – 3N

14 – 3X = -X + 4

$9X - 1 = 25 - 4X$

$14 + 4Y = 8Y + 6$

$-2 + 4X = 13 + 3X$

$7C + 9 = 11C - 31$

$28 - 6A = 4A - 2$

$25 - 2N = 5 + 3N$

$10N - 8 = 3N - 1$

$7 - 4X = 5X - 2$

$5 + 13B = -15 + 3B$

$18 - 5A = A + 3A$

$3 - 4J = 8 + J$

$\frac{Y}{8} + 12 = 22 - 9$

$9 = \frac{18}{N} + 3$

$\frac{32}{B} - 5 = 3$

$20 + \frac{C}{3} = 5\ (5)$

$\frac{H}{9} + 14 = 16$

$19 + \frac{X}{2} = 8 + 5$

$2 - 2X = 17 + X$

APPENDIX B
FOLLOW-UP ASSESSMENT

Name: ______________________________ Date: ______________________

Year in School: ____________________

Teacher's last name: ________________

Solve for the variable. Circle your answer and write legibly.

ex. $5N = 15$
$N = 15/5$
$N = 3$

$6 - 2X = X - 6$

$28 - 6A = 4A - 2$

$25 - 2N = 5 + 3N$

$10N - 8 = 3N - 1$

$7 - 4X = 5X - 2$

$3 - 4J = 8 + J$

$\frac{Y}{8} + 12 = 22 - 9$

$9 = \frac{18}{N} + 3$

$\frac{32}{B} - 5 = 3$

$20 + \frac{C}{3} = 5\ (5)$

$\frac{H}{9} + 14 = 16$

$5X - 8 = 2X - 7$

$-8X + 7 = 22 - 3X$

$19 + \frac{X}{2} = 8 + 5$

$2 - 2X = 17 + X$

$7 - 3X = 4X + 14$

$6Y - 3 = 3Y - 9$

$5B + 8 = 3B + 26$

$5Y + 9 = 8Y - 15$

$-4N + 12 = 21 - 3N$

$14 - 3X = -X + 4$

$9X - 1 = 25 - 4X$

$14 + 4Y = 8Y + 6$

$-2 + 4X = 13 + 3X$

$7C + 9 = 11C - 31$

$5 + 13B = -15 + 3B$

$18 - 5A = A + 3A$

APPENDIX C
TREATMENT FIDELITY CHECKLIST

Teacher Checklist

Teacher Name: _______________

Date: _____________

| | Concrete | | Representational | | Abstract | |
|---|---|---|---|---|---|---|
| Teacher components: | Included | Not Included | included | not included | Included | Not Included |
| Advanced Organizer | | | | | | |
| Description of activity | | | | | | |
| Model | | | | | | |
| Guided Practice | | | | | | |
| Independent Practice | | | | | | |
| One-minute probe | | | | | | |

Pace of instruction (subjective)

Concrete: Good _____ Hurried _____ Too leisurely _____

Representational: Good _____ Hurried _____ Too leisurely _____

Abstract: Good _____ Hurried _____ Too leisurely _____

Percent of components included = ____ / 21 * 100 = ____ %

# APPENDIX D
# CASE STUDY PILOT FOR LARGE N STUDY

Two subjects in Alachua County public schools similar in age and grade level, previous algebra experience, and academic disability label participated in the pilot study of the CRA sequence of instruction in algebra. An ABAB alternating treatment design was used to determine the effectiveness of the CRA sequence of instruction describe above. Students were selected based upon poor performance the year before in algebra. The instructional steps used in the control may well be a treatment themselves. Most students are merely shown examples of problems on an overhead or blackboard then asked to answer a series of problems similarly to what was demonstrated. The instructional components used in this study included an advanced organizer to explain relevance and a guided practice step to work with the student for the teacher to learn about the student's difficulties and develop proper techniques. For this reason it is possible that students may react positively to these steps as well as the treatment that incorporates concrete manipulatives and representational pictures.

Initial baseline was the battery of pretests. Following the baseline and selection of students, students were randomly assigned to either the ABAB or the BABA design. In each case, the student received 3 days of either the treatment or control condition. The treatment condition applied 1 day of concrete instruction followed by a probe, the next day of representational instruction followed by a probe, and the final day of abstract equations followed by a probe and a test for generalization of the steps learned. The

control condition received 3 days of abstract instruction, each day followed by a probe, and the final day also followed by the generalization probe. After the first 3 days of instruction, the student received instruction on the same algebra concept in the alternative instructional sequence to what they previously received. After the 6 days of instruction, a new algebra concept was taught. The student who received treatment first for one concept started with the alternative sequence for the other concept. The final 6 days of instruction followed the same pattern that the first 6 days of instruction did, but with the new concept. The learning sheets used as the guide to ensure treatment integrity were equal for both students regardless of whether the student was receiving treatment or control.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| S1 | pre | A+ | A+ | A+ | B+ | B+ | B+ | B- | B- | B- | A- | A- | A- | Post |
| S1 | pre | B+ | B+ | B+ | A+ | A+ | A+ | A- | A- | A- | B- | B- | B- | Post |

Baseline Establish lack of trend and stability

Treatment CRA scripted lessons versus A only scripted lessons

Posttreatment Three week follow-up to establish retention

Results Visual inspection comparing the growth and acquisition of CRA (treatment A) versus A only (treatment B).

Conclusions: While the reversal design is not effective at displaying differences across treatment and control, evidence of the power of the CRA treatment was still evident. For each individual task, solving single variable equations with addition or with subtraction may be easily argued that the increases in the performances would have occurred with either treatment or control. The generalization charts were more

convincing. While Sharie had some degree of steady growth in understanding, she was provided the treatment first. She remarked to the tutor how she had never understood the lessons before she was able to apply algebraic knowledge hands-on. This understanding may have carried over into the traditional style of instruction. This means that even though the treatment was not being repeated, Sharie's learning could not be reversed. She provided further evidence of this carryover effect by writing pictorial representations on probes and assignments given under the control conditions. For Mark, however, the order of instruction was reversed. Mark may have had progress obtained from the first work with traditional instruction using direct instruction; his learning could not be generalized across more difficult equations (adding a coefficient other than 1). In fact, on the first generalization probe following 3 days of instruction on solving for single variables using addition, he obtained more incorrect digits and still no correct digits. After the CRA treatment, he improved. After the treatment on solving for single variable equations using addition, he obtained eight correct digits and six fewer incorrect digits. While he was not at mastery for this task, after learning how to solve single variable equations using the CRA sequence, he improved his scores on the assessment measuring correctness of more complex problems.

APPENDIX E
SCRIPTED LESSON EXAMPLES

# CRA Delivery Sheet

**Ideas for material distribution**
1) **Individual paper bags for each student to decorate to establish ownership**
2) **Keep extras aside for last-minute emergencies**
3) **Replace toothpicks with small popsicle sticks from Walmart craft section if concerned. Each bag of 150 covers approximately seven students.**

**Probe use**
1) **Start on a new problem every 2-minute session.**
2) **Be careful that the probe being used matches the corresponding lesson.**
3) **If time does not permit graphing number correct with number incorrect, then save sheets in a separate folder for the study team to analyze and grade later.**

**Pre-post-followup testing**
1) **Do not set a time limit for statistical validity reasons**
2) **Warn students that the test is not for a grade but for them to do their very best.**
3) **Encourage students to try to work through problems that seem difficult. Every effort counts, especially when we are assessing error patterns.**

**Concrete stage:**
*) **Encourage material use even when students say they do not need it.**

**Representational stage:**
*) **Encourage students to connect their drawings to what they remember about the materials**

**Abstract stage:**
*) **Students may use representations at this stage, but the purpose of abstract stage is to build a smooth flow of knowledge.**

**Learning Sheets, probes, tests:**
***** Save all sheets with student's name on each. This is very important for us. *****

**Absentees**

***) If at all possible, allow student to catch up by forcing them through the sheets and probes. Otherwise, they would be dropped from the overall study.**

**Student and Parent Consent Forms**

***) Hold onto all of these for us to collect. We may not use student data or examine student file scores otherwise.**

**Thank you. This program is among many of the developing algebra series to help students in algebra who otherwise may have struggled.**
**Brad Witzel**
**813-253-3333 x3423**
**727-546-6180**

Teaching script for lesson 1 **Concrete**

Lesson One: Reducing terms in an equation

Materials: sticks for place value 10s, cups for coefficients, toothpicks for ones, strings for equal signs, letter cards for variables, multi-colored cards for operations and strips of paper for division. Students will also have a learning sheet and progress chart.

Advance Organizer:

**Today we are going to start a new program to prepare for algebra. By the time we are complete with this series of work, you will be able to solve for variables through a series of steps. Solving for variables is very important because it helps us discover unknowns. Before we start our first lesson, lets learn some of the items you are about to receive.** (Hand out each material immediately following its introduction). **This is a Popsicle stick.** (Hold up Popsicle stick). **This represents the number 10. If I hold up one stick I am showing the number 10. What am I showing when I hold up two Popsicle sticks?** (Elicit response.) (Hand out Popsicle sticks.) **This is a toothpick.** (Hold up toothpick.) **This represents the number one. If I hold up 7 toothpicks, what number am I representing?** (Elicit response.) (Try more questions.) **Remember, toothpicks are for math use only. They are to be used for the equations we will go over together and are not to be used for pointing to items or touching others. This is a cup.** (Hold up cup.) **The cup will come before or after variables. This string is fun.** (Hold up string.) **We will actually be using the string to split the equation across the equal sign. The multicolored cards are for operations.** (Hold up +/- card.) **The green side is the positive or plus side. What do you think is on the other side that is colored red?** (Elicit response.) **That is correct. The other side is a minus or negative side. The last piece is a strip of paper.** (Hold up paper.) **This paper will be used when division is performed. Now that you have every piece of the materials, get ready. Today we will start with reducing terms in an equation. You will use your entire desk so make sure to have out only your learning sheet, materials, and pencil.**

Demonstrate and Model

Let's look at problem (a) on the learning sheet. **This year I earned some money. I earned two payments for mowing the lawn.** (Show a N for mowing the lawn and two cups for two payments.) **$10 in gifts from my parents** (show a stick to represent 10), **one more payment for mowing another lawn** (show another N and one cup), **and I had to give $5 to my little sister** (hold up 5 toothpicks). **I need to string these items together to form an equation. The N represents mowing the lawn and the cups represent how many times. Since I earned 10 in gifts from my parents I should add them to the two cups of N.** (Hold up the plus sign on the green side of the colored cards.) **Since I earned the next payment for the other lawn mowing, I should add the one cup of N to 2N + 10.** (Hold up another plus sign.) **Now, I had to give my little sister the 5 dollars so I need to subtract 5.** (Hold up a red minus sign. Draw how the diagram should look on the board for students to see.) **Hmmm, which ones of these may I add together? Since the cups share the variable N, I should be able to add those together. Let's see, there is no plus or minus in front of the two cups of N so I know it must be a positive number. There is a positive in front of the 1 cup of N. These are both positive so I add them together to form three cups of N.** (Write three circles and a N on the board.) **The other numbers do not have a variable attached so I can combine those. The 10 is a positive and the 5 has a negative attached to it. I can't break the stick to subtract so I better trade that for 10 toothpicks. After all, this stick is worth 10 toothpicks.** (Count off ten toothpicks aloud and place the stick aside.) **So I need to subtract 5 from the positive 10. Well let's remove them at the same time. I will take one from each group.** (Take away one from each group until one pile has 0.) **I have left a positive 5 and a minus 0 so I can push aside the minus 0. I have left 3 cups of N and 5 toothpicks. This translates to 3N + 5.** (Write the answer on the board.) **Write down 3N + 5 on your papers in part (a). For part (b) I want you to follow along.**

**I have 5 cups of X.** (Hold up five cups and an X card.) **Next, I have a plus 5.** (Hold up a plus card and 5 toothpicks. Check to make sure the students have the same display on their desks.) **Now I have to take away 6 toothpicks.** (Show the students a minus sign and 6 toothpicks. Make sure students are with you.) **Which of these may I combine. I cannot combine the 5X and the toothpicks because the variable cannot be mixed with other variables or numbers alone. So I may combine the 5 and minus 6. Let's take one from each group at the same time.** (Take away toothpicks from groups simultaneously. You will have 0 from the group of five and 1 toothpick next to the minus sign. Make sure students are duplicating your action.) **Hmm. I have one toothpick next to the minus sign. I guess this means that I have a negative one.** (Remove extraneous materials.) **So I only have left 5 cups of N and a minus 1 toothpick. This means I have 5N – 1 left.** (Write 5N –1 on the board.) **Write down 5N –1 on your papers next to question (b).**

Guided Practice

**Now let's try some together. For question (c) what do we lay down to represent 3Y?** (Elicit answer.) **Good, we lay down 3 cups and a Y card. What do we lay down to represent the plus sign.** (Elicit answer.) **Good, a green plus card. Now how do we represent a Y?** (Elicit answer.) **Good a Y card, but also one cup. It is only**

**1 Y is there is no other number in front of the variable. Now, how do we represent the plus again?** (Elicit answer.) **Good, a green plus card. The five is represented how?** (Elicit answer). **Good, five toothpicks. Now, what may we combine with this equation?** (Elicit answer.) **That is correct. We can add the 3 cups of Y to the one cup of Y. How many cups of Y do we end up with?** (Elicit answer.) **Correct! We end up with 4 cups of Y. Can we add the 5 toothpicks to anything?** (Elicit answer.) **Correct. We cannot add the toothpicks to anything but other toothpicks or sticks. What do we have in the end?** (Elicit answer.) **Good. We have 4Y + 5. Write down the answer on your sheet.** (Repeat the guided practice for (d), (e), and (f) with decreasing teacher input. By (f), students should be setting the equation up n their own. Circulate around and have students help each other if needed.)

Independent Practice

**Try the next problems on your own. When you come to the word problems, do your best to create the equation and then solve the problem.**

**Probe**

**Now that you have shown how well you understand this concept, let's try a few for speed. Answer as many as possible in 2 minutes. We will graph our answers the following class.**

Teaching script for lesson 4: **Concrete**
Solving for single variables basic operations.

Materials: sticks for place value 10s, cups for coefficients, toothpicks for ones, strings for equal signs, letter cards for variables, multi-colored cards for operations and strips of paper for division. Students will also have a learning sheet and progress chart.

Advance Organizer:

**Today we are going to solve for single variables using operations such as adding, subtracting, dividing and multiplying. This will help us understand how to mathematically solve for something we do not know. Before we start, let me review the materials we will use to solve these problems. This is a Popsicle stick.** (Hold up Popsicle stick.) **What does the popsicle stick represent?** (Elicit response.) **Good, this represents the number 10. If I hold up one stick I am showing the number 10.** (Hold up toothpick.) **What do toothpicks represent?** (Elicit response.) **Good, this represents the number one. If I hold up 4 toothpicks, what number am I representing?** (Elicit response. Try more questions.) **Remember, toothpicks are for math use only. They are to be used for the equations we will go over together and are not to be used for pointing to items or touching others. This is a cup.** (Hold up cup.) **The cup will come before or after variables. This string is fun.** (Hold up string.) **We will actually be using the string to split the equation across the equal sign. The multicolored cards are for operations.** (Hold up +/- card.) **What does the green side represent?** (Elicit response.) **Right, the green side is the positive or plus side. What does the red side represent?** (Elicit response.) **That is correct. The other side is a**

**minus or negative side. The last piece is a strip of paper.** (Hold up paper.) **This paper will be used when division is performed. Now that you have every piece of the materials, get ready. Today we will start with reducing terms in an equation. You will use your entire desk so make sure to have out only your learning sheet, materials, and pencil. Now let's begin.** (Hand out learning sheets and materials).

Demonstrate/Model

**Now, my friend lent me some money to buy a CD, but I forgot how much.** (Place a cup and an X on a cellophane plate in a visible spot for the class to see.) **After I leave the store, I read the receipt. The receipt says I spent 10 dollars on the CD.** (Show the class the popsicle stick.) **Now, since I *spent* the money, am I going to add the money to what I was lent or take away? ... Yes, take away.** (Place a minus sign after the X and the 10 after the minus sign.) **Now in my pocket I have three \$1 bills left. Now since I have \$3 left does that go before or after an equals sign? ... Right, after the string.** (Lay the string and the 3 after that.) **The equation now reads X-10=3. This would be a lot easier to solve if the X was all alone on one side of the equation. Let's discuss the mnemonic ISOLATE. The I tells me to identify the variable and the equal sign.** (Point to the variable and then the equals sign. Draw a squiggly line down the paper under the equal sign for dividing the problem.) **The S tells me to do calculations to isolate the variable. This means I have to move the 10. To put the 10 on the other side, I need to do something. The O in ISOLATE tells me to organize the equation to equal out what I have added. So if I do something to the equation, I have to do it to both sides of the equals sign. Hmm, if the 10 reads a minus 10, then I can make that 0 by adding 10 to it. The L tells us to let the calculations fly. The A tells us to answer the calculations around the variable first.** (Place a + 10 on both sides of the equation under the equation.) **What happens to +10 –10? ... That's right, they equal 0. Good, the variable is isolated. Now the T tells us to total the other side. So, how about 3 + 10? They equal 13. I have left X + 0 = 13. That means X must equal 13. My friend gave me \$13. The E asks us to evaluate if this makes sense. Well, 13-10 does equal 3, so our work is correct.** (Go through the next modeling questions using think alouds.)

Guided Practice

**Now it is your turn to try a few.** (Work through the next two guided practice problems step by step by having students represent the equations. Do not work ahead. Let the students state what happens next in each step for solving the equations. Circulate about the room to make sure each student is working at the same pace and check for student movement of the materials. For the last two problems, allow students to work more on their own with only minimal teacher input. Write each problem as students are working and use the representational diagrams if students are unable to see.)

Independent Practice

**Let's try these problems on your own. As you notice, there are word problems. Do the best you can and try to complete each problem on your own.**

Probe: Start on the second line of the sheet and do as many as you can on 1 minute.

Teaching script for lesson 5: **Representational**
Solving for single variables by addition.

Materials: Presentational device (either overhead or blackboard), lesson worksheet

Advanced Organizer

(Write T- 14 = 12). **Last time we worked on solving for a single variable using hands-on materials. This time we are taking a step further. This time we are going to draw out the parts to the problem we are given in the same manner we used the materials.**

Demonstrate/Model

**For example, what did we use to represent the variable? ... Right. We used a paper letter. This time, instead of using a paper letter, we will represent variables by simply writing them down.** (Write the X under the X in the equation.) **What did the cup stand for? ... Groups** (or coefficients) **is correct. To represent groups we will use an empty circle next to our variable.** (Write a circle next to the X in the equation.) **What did we use to represent the numbers? In this case the 14. ... Popsicle stick and toothpicks is correct. Instead of sticks though, we are going to use tally marks. One small diagonal mark equals one** (write a small dash), **but to represent ten we will use a long straight mark** (Write the long dash.) **In this problem, we must represent fourteen so what do I write?** (Erase the previous dashes.) **... Correct, I make one long mark and four tally marks. Now to represent the minus sign, I am just to make a minus sign and to represent the equals sign, I will draw a squiggly line overtop the equals sign, just as we had on our desks** (Draw lines.) **To represent the 12 then, what do we write? ... One long mark and 2 tally marks is correct.** (Write the marks.) **Now we could solve this problem.**

**I know the mnemonic "ISOLATE" means I am solving for the X.** (Go through steps of the mnemonic slowly and thinking about each step to the equation aloud.) **Let's try the next one.** (Write down 6=Y-23 and go through the same steps asking students just to watch. For the final two modeling practices, ask students to work with you as you go through the problem.)

Guided Practice

**Now that you have seen me do this, lets try some together. How about 2 = -15 + H? Let's go.** (I ) **What are we solving for? ... Right, the H. Now locate the equals sign by making a squiggly line down the equals sign. Good. We are solving for the H and how many groups of H are there? In other words, are there any numbers next to Z NOT separated by a plus or minus sign? ... No, there aren't. So how many groups of H are there?** (No, not zero.) **... Right, there is one group of H. So I place a circle next to Z. I see a two and a -15. This means I write what underneath them? ... Tally marks are good. Write 2 tally marks under the 2 and one long mark and 5 tallys under the -15. Since the –15 has a minus in front of it, let's place a minus there. I also see a plus sign, what goes there?** (Elicit response.) **Good, a minus sign. (Go through the ISOLATE steps).**

Independent Practice

**Do the best you can on the independent practice section and work on the problem solving section.**

Probe

Hand out probe and time them for 2 minutes.

Abstract Lessons

All Comparison lessons

CRA lessons 3, 6, 7, 10, 11, 14, 15, 18, 19

Advanced Organizer

State why the operation is important for the students to learn. Remind them that not all math equations for discovering lost numbers is easy so we must continue practicing. In these lessons, students are encouraged to work with numbers only. Students with representational strategies *may* use them to fall back on.

Describe and Model.

Use think aloud strategies for each problem in this section. Encourage students to watch closely as each lesson adds a new dimension to solving for unknowns.

Guided Practice

Start by going step by step through the problem with them. By the last problem in this section, students should be writing down the next step faster than you can cover it.

Independent Practice

Start them off by stating that they are to do their best and try not to ask questions from the teacher, but to work out the problem on their own. Since word problems are not fully covered at this stage, ask them to work them through logically.

"Make your own" word problems should be encouraged for higher achieving students to work independently. Other students may want to discuss ideas with each other or use previous problems for examples.

Probes

Start with a new section of the sheet and have students complete as many as possible.

## APPENDIX F
## LEARNING WORKSHEETS

1. Reducing by combining like terms

Lesson Sheet 1 Concrete

Describe and Model

a) $2N + 10 + N - 5$

b) $5X+5-6$

Guided Practice

c) $3Y + Y +5$

d) $Y + Y + 2$

e) $\frac{3}{N} - 12 + 6$

f) $8X - 3X + 14$

Independent Practice

g) $N + N + 12$

h) $-X + 12 + 5$

i) $5 + 2N + 7$

j) $3X + Y + 2X$

k) $-N -2N + 3$

l) $Y + 3N - Y$

Problem Solving

m) You have 5 large coins (5Y) and (+) 2 small coins (2N). You find 1 more large coin (+1Y). Set up the equation and show how many small and large coins you have.

n) In PE, you have 3 slow runners (3X) and (+) 2 fast runners (2Y). 2 more fast runners (+1Y) join your class. Then 1 slow runner leaves (-1X) the class. Set up the equation and combine to show how many slow and fast runners remain.

Learning Sheet 2 Representations

Describe/Model

a) $5N - \frac{2}{X} + N - 12$

b) $9 - 3X + 8 + X$

Guided Practice

c) $-6 + T + 4T + 2$

d) $2N - 3M + 12 + M$

e) $7H + 3V - 6H - 2$

f) $T - 3T + 2T + 2$

Independent Practice

g) $4U + 5U - W + 2W$

h) $6P + 4 - 3P + 7$

i) $\frac{W}{3} + 4 - 12 + N$

j) $2L + 6W - 2L + 13$

k) $3X + 4Y + 14 - 4X$

l) $Y + 23 - 2W - 12$

Problem Solving

m) You can carry only so many books. You are carrying 2 big books (2B) and 4 small books (4L) along with 4 pieces of paper (4). You next teacher gives you 2 more big books (2B) to carry home. Set up the equation to show how many books and paper you have to carry.

n) You are helping make a cake at home. The cake takes 2 cups of flour (2F) and 1 cup of sugar (1X). The icing takes 2 cups of sugar (2X) and 1 cup of water (1W). What does the cake take total of these ingredients? Set up the equation to solve the problem.

Learning Sheet 3 Abstract

Describe/Model

a) 8 – F – 12X * 2

b) 9 / 3 + 8X + X + 1

Guided Practice

c) –N + 4M + 2W + 2N

d) $\frac{24Y}{6} + 6 + Y$

e) $\frac{7}{H} + 3K - 2 / 4 - K$

f) –5N + 5 – 5N + 10 + C

Independent Practice

g) 4U * 5 – W + 2W

h) –3P + 7 + P – 12

i) $\frac{12W}{3} + 4 - 12$

j) 2W + 6 – 2K + 2K

k) 6X * 4 + 14 – 4X

l) 2B – 2W – 12 + 10

Problem Solving

m) Your uncle Leon brought 4 red presents and 2 blue presents to the party. Your little brother then took two blue presents. Aunt Patty brought 2 more red presents. Set up the equation to figure out how many presents were left.

n) The football team passed the ball for 5 yards and then ran for 4 yards. They passed for 10 yards more and then ran for a loss of 2 yards. Set up the equation to see how many yards were moved by the team according to running and passing.

2. Solving inverse operations (fractional coefficients, add, subtract)

Learning Sheet 4 Concrete

Describe / Model

a) $X - 10 = 3$

b) $13 = X + 5$

c) $\frac{N}{2} = 5$

d) $3Y = 24$

Guided Practice

e) $5 = Y - 3$

f) $7 + N = 8$

g) $3X = 21$

h) $\frac{N}{4} = 9$

Independent Practice

i) $18 = Y + 5$

j) $X - 11 = 2$

k) $9 = 3Y$

l) $\frac{X}{2} = 6$

Problem Solving

m) Warren (X) ran (=) for 5 fewer yards than Mike (Y - 5). Warren ran for 23 yards one game. How many yards did Mike run?

$X = Y - 5$

Let Y = Mike's yards and X = Warren's yards

________ = ________ - ________
Warren Mike fewer yards

n) Today (T), Bill (N) works 2 less (- 2)than he normally does. Today (=) he worked 6 hours. How many hours does Bill usually work?

Let N = hours Bill worked today

The equation: ________ = ________ - ________

________ = ________ - ________
Usual hours today less hours

Learning Sheet 5 Representational

Describe/Model

a) $T - 14 = 12$

b) $6 = Y - 23$

c) $3 = \frac{X}{6}$

d) $8C = 64$

Guided Practice

e) $2 = -15 + H$

f) $\frac{W}{4} = 3$

g) $-8 + F = 2$

j) $14 = 7X$

Independent Practice

k) $P - 6 = 13$

l) $14 = 2 + T$

m) $25 = 5Y$

n) $13 = X - 14$

o) $-3 = \frac{N}{9}$

p) $\frac{P}{6} = 7$

Problem Solving

q) Carl (C) had a large number of chocolates. He split them among 5 friends. After he split them, each friend had 10 chocolates. How many did he have total? Set up the equation and answer.

r) Suzi was loaned money by her mom, but she forgot how much. In her pocket she read a store receipt that said she spent 10 dollars for a CD and had 10 dollars change. How much did her mother loan her? Set up the equation and solve.

Learning Sheet 6 Abstract

Describe/Model

a) $8 = Y - 21$

b) $\frac{W}{4} = 11$

Guided Practice

c) $2 = 9 + P$

d) $14 = 7X$

Independent Practice

e) $P - 18 = 8$

f) $54 = 14 + M$

g) $16 = 4F$

h) $11 = -22 + C$

i) $81 = 9D$

j) $8 = \frac{U}{6}$

k) $63 = 9Y$

l) $16 = W - 7$

Problem Solving

m) Al's backpack could carry 5 books. How many backpacks would Al need to carry 25 books? Set up the equation and solve the equation.

n) There are 11 players per team and many teams. If there are 88 players, how many teams are there? Set up the equation and solve.

Learning Sheet 7 Abstract II

Describe/Model

a) $T - 14 = 12$

b) $21 = \frac{M}{8}$

Guided Practice

c) $12 = \frac{X}{8}$

d) $15C = 45$

Independent Practice

e) $X + 8 = 26$

f) $\frac{K}{7} = 6$

g) $19 = X - 3$

h) $64 = 4X$

i) $-2 + Y = 17$

j) $-23 + R = 1$

k) $7 = \frac{N}{4}$

l) $\frac{P}{16} = 3$

Problem Solving

m) Jose had 20 pencils and gave a few away. He had 3 left. How many did he give away? Set up the equation and solve.

n) Write your own problem using a subtraction, an unknown, and pennies to form your equation.

3. Solving when the variable is the divisor or being subtracted

Learning Sheet 8 Concrete

Describe / Model

a) $-N + 10 = 3$

b) $12 = -X - 5$

c) $\frac{10}{X} = 5$

d) $\frac{12}{Y} = 6$

Guided Practice

e) $5 = -X - 3$

f) $7 - N = 8$

g) $\frac{12}{Y} = 4$

h) $\frac{15}{X} = 3$

Independent Practice

i) $8 = -Y + 1$

j) $-X - 10 = 2$

k) $\frac{8}{Y} = 2$

l) $\frac{21}{N} = 3$

Problem Solving

m) A cake has 18 pieces. If the pieces must be divided among a few people and each person received 3 pieces of cake, how many people were there? Set up the equation and solve for the variable.

n) The shop owner has $15. She gives you some for helping her. You see she kept $2. How much money does she give you? Set up the equation and solve for the variable.

Learning Sheet 9 Representational

Describe / Model

a) $10 - N = 3$

b) $12 = -5 - X$

c) $\frac{10}{X} = 5$

d) $6 = \frac{12}{Y}$

Guided Practice

e) $-4 = 13 - C$

f) $-W - 2 = 17$

g) $5 = \frac{25}{Y}$

h) $\frac{27}{M} = 3$

Independent Practice

i) $9 = -Y + 8$

j) $16 - X = 12$

k) $\frac{24}{Y} = 6$

l) $\frac{56}{N} = 7$

Problem Solving

m) You pull out $20 to pay for you haircut. You don't know how much the haircut cost. After you hand the money over (=) the barber hands you $7 change. How much did the haircut?

n) Mr. W. had to divide 8 books among some students (N). Each student had 2 books. How many students were there? Set up the equation and solve.

Learning Sheet 10 Abstract

Describe / Model

a) $\frac{72}{P} = 8$

b) $8 = 15 - H$

Guided Practice

c) $\frac{36}{M} = 12$

d) $-9 = -7 - C$

Independent Practice

d) $9 = \frac{54}{Y}$

f) $-X + 13 = -17$

g) $8 = \frac{32}{Y}$

h) $\frac{49}{R} = 7$

i) $19 = -Y + 18$

j) $-13 - X = 13$

k) $\frac{77}{K} = 11$

l) $24 - P = 22$

Problem Solving

m) 90 of Suzie's closest friends came to her birthday party. She divides them into several teams to play football. If each team has 9 people, how many teams are there? Set up the equation and solve.

n) Last year it rained an unknown amount. It rained 10 inches less this year. This year it rained 28 inches. How many inches did it rain last year? Set the equation and solve.

Learning Sheet 11 Abstract II

Describe / Model

a) $52 = -A + 32$

b) $\frac{54}{X} = 6$

Guided Practice

c) $\frac{42}{M} = 13$

d) $37 = 14 - H$

Independent Practice

d) $3 = \frac{30}{Y}$

f) $-J - 8 = 16$

g) $23 = \frac{23}{G}$

h) $\frac{24}{U} = 8$

i) $9 = -Y + 8$

j) $16 - X = 12$

k) $\frac{48}{P} = 8$

l) $2 - N = 31$

m) $\frac{26}{M} = 2$

n) $19 = -10 - D$

Problem Solving

m) Mrs. Millionaire owned 10 cars and divided them among his daughters. If each daughter ended up with 2 cars, how many daughters were there.

n) Make your own problem using 20 items and the division of an unknown.

4. Solving for a variable when like variables are on the same side

Learning Sheet 12 Concrete

Describe / Model

a) $-N + 2N = 12$

b) $12 = -X - X + 5$

c) $10 + 5 = 3X - X$

d) $2Y + 1Y = \frac{6}{2}$

Guided Practice

e) $5 * 4 = -X - 3$

f) $7 - N = 8$

g) $12 = 4Y - Y + 1Y$

h) $\frac{15}{3} = 3N + 2N$

Independent Practice

i) $7 = -4Y + 3Y$

j) $-X - X - X = 9$

k) $\frac{8}{4} = 2N - N + N$

l) $Y + 3Y = 16$

Problem Solving

m) 1 employee's (E) salary must be added (+) to another 3 employees' (3E) salaries. Their total (=) salaries are \$28. How much does each employee earn? Set up the equation and solve.

n) 3 batteries (3B) are added to the one battery (1B) in Jan's game controller. The total voltage of the batteries is (=) 16 volts. What is the voltage of each battery? Set up and solve the equation.

Learning Sheet 13 Representational

Describe / Model

a) $9 + N + 2N = 3$

b) $3\ (5) = -4 + 3H - H$

c) $10 + 6 = 2Y - 3Y$

d) $\frac{12}{3} = T5 - T$

Guided Practice

e) $-4 = 14 - C\ (3)$

f) $-2W + W - 2 = 17$

g) $2G + G + G2 = \frac{10}{5} + 3$

h) $\frac{2}{2}W + W = 13 + 3$

Independent Practice

i) $P4 - 2P = -12 + 8$

j) $5X - X = 11 + 4 - 1$

k) $\frac{16}{8} + 7 = 4P + 5P$

l) $\frac{12}{6} = 7Y - Y5 + 6$

Problem Solving

m) Chuck brings home 2 paychecks from last week. He adds these to the previous 5 paychecks. He now has made \$84. How much is each paycheck? Set up and solve the equation.

n) Marsha mowed 5 lawns on Saturday and 2 lawns on Monday. She made \$45 on Saturday and \$18 on Monday. How much did she earn for each lawn cut? Set up the equation and solve.

Learning Sheet 14 Abstract

Describe / Model

a) $\frac{72}{8}(5) = -8Y + Y3$

b) $12 - 6 = \frac{12}{M}$

Guided Practice

c) $36 + 12 = 12 - 2W - W$

d) $-9\,(11) = -7C - C4$

Independent Practice

d) $3W\,(3) = \frac{54}{9} + 24$

f) $-X - 4X = -17 + 2$

g) $8W - 4\,(W) = \frac{38}{19} + 3\,(2)$

h) $75 - 3 = 12Y - Y - Y - Y$

i) $9 = -Y - 8Y$

j) $12N - 5N - 5 = 13 + 10$

k) $\frac{4}{V} = 1 + 3$

l) $24K - 12K = 22\,(2) + 4$

Problem Solving

m) Erin brought you some money for your birthday and gave 2 times as many to your sister. She brought \$15 total. How much did she bring to you? Set-up the equation and solve.

n) Build you own problem similar to the one above using 2 unknowns and a known number.

Learning Sheet 15 Abstract II

Describe / Model

a) $\frac{32P}{4} - 2P - 8 = 40$

b) $\frac{15}{C} = \frac{10}{2}$

Guided Practice

c) $\frac{42}{7W} = \frac{24}{4}$

d) $12\,(3) = 3M + 9M$

Independent Practice

d) $7\,(W) - 3W = \frac{38 + 10}{2}$

f) $48 = 8P - 2P + P + 20$

g) $N + 6\,(N) = 35$

h) $\frac{85}{5} = \frac{5Y + 4Y}{9}$

i) $\frac{10}{K} = \frac{35}{7}$

j) $21F - F + 2F - 10 = 34$

k) $\frac{36}{6} = \frac{3\,(8)}{V}$

l) $\frac{34X}{2} - 7X = 30 + 10$

Problem Solving

m) Eric divided the $40 to pay for the movies among his 4 friends. He paid the extra cost for the popcorn and the drinks. The total cost was $56. How much did Eric spend on popcorn and drinks? Set up and solve the equation.

n) Make up your own question similar to the one above and solve.

5. Solving for a variable when like variables are on opposite sides

Learning Sheet 16 Concrete

Describe / Model

a) $X = 2X + 4$

b) $3X - 18 = X$

c) $2N = 5N - 15$

d) $3Y = 25 - 2Y$

Guided Practice

e) $Y = 2Y - 3$

f) $6 + N = -N$

g) $3X = 12 - X$

h) $N = 5N + 8$

Independent Practice

i) $18 - Y = Y$

j) $X - 11 = 2X$

k) $2Y + 9 = 3Y$

l) $X = 6 - 2X$

Problem Solving

m) Tom had 20 dollars (20) added (+) to his usual pay (X). The total amount (=) was 2 times what he is normally paid (2X). What is his normal pay? Set up the equation and solve.

n) Carrie had 12 dollars (12). Some money was taken away (- Y). She ended up (=) with the same amount that was taken away from her (Y). Set up the equation and solve.

Learning Sheet 17 Representational

Describe/Model

a) $T - 14 = 3T$

b) $6 - Y = Y - 24$

c) $3 - 3X = X - 9$

d) $5C - C = C + 18$

Guided Practice

e) $2 + H = -19 + 4H$

f) $7W = 3 + 4W$

g) $-8 + F = 2F + 11$

h) $14 + V = 3V + 2$

Independent Practice

i) $-P - 7 = 13 + 9P$

j) $29 - 2T = 2 + T$

k) $25 + Y = 5Y - 3$

l) $13 + 2X = 5X - 2$

Problem Solving

m) A CD (N) and 15 dollars (+ 15) cost (=) the same as 2 CDs (2N) and 3 dollars. How much does one CD cost? Set up the equation and solve.

Learning Sheet 18 Abstract

Describe/Model

a) $8 + 4Y = Y - 22$

b) $2K + 3K = 11 + 2 + 4K$

Guided Practice

c) $5 - 4W = -9 + 3W$

d) $-7X - 13 = 7X + 1$

Independent Practice

e) $P - 18 = 2P - 26$

f) $54 + 6M = 12 - M$

g) $8F + 16 = 2F - 26$

h) $11 - 25C = - 22 - 14C$

i) $40 - 2D = -41 + 7D$

j) $8U + U - 17 = U + 47$

k) $35 + 2Y = 9Y - 14$

l) $19W + 16 = 7W - 8$

Problem Solving

m) Two brothers were given the equal amounts of money. Brandon bought one videogame for himself and had 12 dollars left. Taylor bought 2 videogames and 0 dollars left. Since both brothers had the same amount of money, how much did each videogame cost? Set up the equation and solve.

Learning Sheet 19 Abstract II

Describe/Model

a) $\frac{T}{3} - 3 = T + 1$

b) $4 + M = \frac{M}{2} + 1$

Guided Practice

c) $12 - X = \frac{X}{3} - 8$

d) $15C - 12 = 13 + 10C$

Independent Practice

e) $X + 8 = 26$

f) $\frac{K}{7} + 4 = K - 2$

g) $19 = X - 3$

h) $64 = 4X$

i) $-2 + Y = 17$

j) $-23 + R = 1$

k) $2N - 8 = \frac{N}{4} - 1$

l) $\frac{P}{2} + 2 = 3P + 12$

m) $-3B + B - 1 = B + 8$

n) $2N + 5N - 25 = 7 + N + 16$

Problem Solving

o) Using equations you learned during these lessons, write your own problem and set up the equation to go with it.

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## BIOGRAPHICAL SKETCH

Brad Witzel was born in Fairfax, Virginia, on May 26, 1972. Unsatisfied with his years of high school at a math and science magnet school, he researched stress levels of students with different levels of giftedness. Although some administrators were not pleased with the implications of the investigation, Brad's points were heard at the state level. He graduated from high school in 1990. The following year he attended Frostburg State University in Maryland. He participated in soccer, acting, student education, and physics societies while maintaining a 3.9 GPA. The next year he transferred to James Madison University in Virginia where, under the guidance of Dr. Esther Minskoff, he began to study students with special needs. He created the position of JMU volunteer coordinator for Special Olympics and hosted dinners and outings for citizens with disabilities. He graduated with his B.S. in psychology and minor in special education in 1994.

Brad's first teaching position was at J.E.B. Stuart High School in Falls Church, Virginia. He was hired as a math and science teacher, but after repeated behavioral successes with students with challenging behaviors, he taught nine different subjects in his first three years. He was highly decorated by fellow staff and held several leadership positions within his first two years. Under the leadership of Mrs. Marcy McGahee he began to work on student-led IEPs. He spoke with Mrs. McGahee and students at local and national meetings regarding the techniques. When he moved to Pinellas County, Florida, he continued his work helping students with disabilities by working with

students with severe and profound disabilities at Nina Harris Exceptional Student Education Center.

In 1998 he entered the University of Florida to pursue his master's degree in special education. He continued his work on student-led IEPs and began work on motivation, which he continues to speak on at local conferences. He earned his M.Ed. in 1999. After being accepting into the doctoral program, he began working with Dr. Cecil Mercer on improving secondary school level math instruction and curriculum. He recently developed a CRA algebra model that is designed to help students learn basic equations immediately and concurrently prepares students for more abstract and complex instruction. His work has been well received by classroom teachers at local schools and national conferences.

While working on his dissertation in 2000, Brad was hired as a full-time instructor on a one-year appointment in the Department of Education at the University of Tampa. Since then, he has served as a research assistant and has divided his time between teaching special education courses to undergraduates at the University of Florida and the University of South Florida.

I certify that I have read this study and that, in my opinion, it conforms to the acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Cecil D. Mercer

Cecil D. Mercer, Chair
Distinguished Professor of Special Education

I certify that I have read this study and that, in my opinion, it conforms to the acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Maureen Conroy

Maureen Conroy
Associate Professor of Special Education

I certify that I have read this study and that, in my opinion, it conforms to the acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Nancy L. Corbett

Nancy L. Corbett
Assistant Scholar of Special Education

I certify that I have read this study and that, in my opinion, it conforms to the acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

M. David Miller

M. David Miller
Professor of Educational Psychology

This dissertation was submitted to the Graduate Faculty of the College of Education and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 2001

Dean, College of Education

Dean, Graduate School